Nonlinear Anisotropic Degenerate Parabolic-Hyperbolic Equations with Stochastic Forcing

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Abstract

We are concerned with nonlinear anisotropic degenerate parabolic-hyperbolic equations with stochastic forcing, which are heterogeneous (*i.e.*, not space-translational invariant). A unified framework is established for the continuous dependence estimates, fractional BV regularity estimates, and well-posedness for stochastic kinetic solutions of the nonlinear stochastic degenerate parabolic-hyperbolic equation. In particular, we establish the well-posedness of the nonlinear stochastic equation in $L^p \cap N^{\kappa,1}$ for $p \in [1,\infty)$ and the κ–Nikolskii space $N^{\kappa,1}$ with $\kappa \in (0,1]$, and the L^1–continuous dependence of the stochastic kinetic solutions not only on the initial data, but also on the degenerate diffusion matrix function, the flux function, and the multiplicative noise function involved in the nonlinear equation.

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1. Introduction

We are concerned with the continuous dependence of stochastic kinetic solutions of the Cauchy problem for the nonlinear anisotropic degenerate parabolic-

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hyperbolic equations with stochastic forcing:

$$\partial_t u + \nabla \cdot \mathbf{F}(u, \mathbf{x}) = \nabla \cdot (\mathbf{A}(u)\nabla u) + \sigma(u)\dot{W} \qquad \text{for } \mathbf{x} \in \mathbb{T}^d, \tag{1.1}$$

and initial data:

$$u|_{t=0} = u_0(\mathbf{x}), \tag{1.2}$$

where $\mathbf{A}(u)$ is a positive semi-definite matrix function so that there exists a positive semi-definite matrix $\boldsymbol{\alpha}$ with $\mathbf{A}(u) = \boldsymbol{\alpha}(u)\boldsymbol{\alpha}(u)^\top$, the flux function $\mathbf{F}(u, \mathbf{x}) = (F^1, F^2, \cdots, F^d)(u, \mathbf{x})$ is heterogeneous (depending on the space variable $\mathbf{x}$), and $\sigma(u)$ is a multiplicative noise function. In the noise term, $W = W(t)$ is a standard (one-dimensional) Brownian motion on the abstract stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.

In this paper, we first develop a unified framework for the continuous dependence estimates on not only the initial data $u_0(\mathbf{x})$ but also the diffusion matrix $\mathbf{A}(u)$, the flux function $\mathbf{F}(u, \mathbf{x})$, and the multiplicative noise function $\sigma(u)$. Then we derive from this continuous dependence framework to obtain both an L^1–stability property and a fractional BV estimate, *i.e.*, a Nikolskii semi-norm estimate defined by (1.5) below, for stochastic kinetic solutions. The motivation for such a study is three-fold: First, equation (1.1) is heterogeneous (*i.e.*, not space-translational invariant) so that the BV-in-space estimate of solutions in terms of the BV initial data does not follow directly from the L^1–stability of solutions, which is different from the space-translational invariant case as treated in Chen–Ding–Karlsen [5]. In fact, the BV-in-space estimate can be obtained only in the special case that $D_{\mathbf{x}} \cdot \mathbf{F}(u, \mathbf{x}) := \sum_{j=1}^{d} F^j_{x_j}(u, \mathbf{x})$ is Lipschitz in its spatial argument $\mathbf{x}$; in general, only a fractional BV-in-space bound (*i.e.*, bounded in the Nikolskii semi-norm) can be obtained, which depends on the Hölder norm of $D_{\mathbf{x}} \cdot \mathbf{F}(u, \mathbf{x})$ in $\mathbf{x}$, as observed in this paper. Second, we carry out our analysis directly from the definition of stochastic kinetic solutions, which is independent of the choices of approximate solutions, different from [5]. Most importantly, we provide a uniform treatment for the L^1–continuous dependence estimates, fractional BV regularity estimates, and well-posedness for stochastic kinetic solutions. For the deterministic case, similar stability problems have been analyzed; see [6, 24] and the references cited therein.

For nonlinear stochastic hyperbolic balance laws:

$$\partial_t u + \nabla \cdot \mathbf{F}(u) = \sigma(u)\dot{W}, \tag{1.3}$$

the L^1–continuous dependence estimates on the flux function $\mathbf{F}(u)$, the noise function $\sigma(u)$, and the initial data $u_0(\mathbf{x})$ have been established in Chen–Ding–Karlsen [5], based on the earlier work of Feng–Nualart [17] on the well-posedness for (1.3). In [17], the existence of strong stochastic entropy solutions, which involve a non-adapted stochastic integral, is achieved by the compensated compactness framework in Chen–Lu [7] for $d = 1$. In Chen–Ding–Karlsen [5], this restriction ($d = 1$) is first removed by combining the BV-estimate they developed with the L^1–contraction estimate of the BV solutions so that the multidimensional case $d \geq 2$ can be handled. It is observed in Karlsen–Storrøsten [23] that there are other ways to achieve the well-posedness to capture the noise-noise interaction in the comparison between two solutions, without resorting to the *prima facie* contrived notion of strong stochastic entropy solutions proposed in [17]. One of them is the kinetic formulation approach that has been carried out in Debussche–Vovelle [14] for (1.3), in which the notion of strong entropy solutions can be avoided via introducing the kinetic defect measure; in this approach, by linearizing the equation via the introduction of a new kinetic variable, the interaction in certain cross terms involving the noise can be handled by the use of the defect measure, instead of the direct integration (see also [15]). In Bauzet–Vallet–Wittbold [2], the formulation of strong entropy solutions is avoided by comparing the stochastic entropy solution directly to the corresponding vanishing viscosity solution. Furthermore, in [23], the Kruzhkov entropy condition is modified to compare a solution to a general Malliavin differentiable variable (instead of a constant), by using an anticipating Itô formula; the vanishing viscosity solution is shown to be Malliavin differentiable, and the framework in [2] is used, for which indicates where the notion of strong stochastic entropy solutions in [17] may arise (*cf.* Remark 5.1 in [23]). It would be interesting to study underlying theoretical connections between the kinetic formulation approach and the Malliavin calculus approach.

It bears pointing out that, in the very specific context of continuous dependence for (1.1), the deterministic and stochastic theories diverge, and we encounter difficulties and structures peculiar to stochastic balance laws. In particular, the Itô correction difference may prevent an account of continuous dependence with the forcing terms involving on the solution itself, in addition to the spatial or temporal variables. However, it is still possible to consider the problem as we do here for which the flux depends on the spatial variable directly, which may have applications in considering stochastic balance laws on manifolds [19], where a connection is spatially dependent, and the kinetic formulation of the equation is more intricate.

Other variations on the well-posedness theory of balance laws have been considered. The most prominent of these arise with conservative Stratonovich noises, in which the noise takes the divergence form. These are of some interest in physical systems as they arise from the perturbation of characteristics [18]. In particular, Fehrman–Gess [16] investigated the well-posedness and continuous dependence of the stochastic degenerate parabolic equation of porous medium type:

$$\partial_t u + \nabla A(\mathbf{x}, u) \circ \mathrm{d}z_t = \Delta(|u|^{m-1}u),$$

including the fast diffusion case $m < 1$, where z_t is a geometric rough path, which includes the case that z_t is a finite-dimensional Brownian motion. This builds on the results collected in [1] for the stochastic PDEs of this form. Also see [8, 14] for the existence of invariant measures for nonlinear conservation laws driven by stochastic forcing.

This paper consists of eight sections. In §2, we introduce the notion of stochastic kinetic solutions in a *divergence form* for (1.1). In §3, we develop a general framework for the continuous dependence estimates of the stochastic kinetic solutions. In §4, we employ the framework in §3 to establish the L^1–stability of stochastic kinetic solutions of equation (1.1). In §5, we employ the framework to derive the fractional BV estimate (*i.e.*, the Nikolskii semi-norm estimate). Using the fractional BV estimate in §5, we complete the L^1–continuous dependence estimate in §6. In §7, we establish the existence of stochastic kinetic solutions. In §8, we derive a temporal fractional BV estimate of stochastic kinetic solutions.

Before we proceed further, we address two notational points: First, we denote ∇ the material derivative, and ∇^i the material derivative in the x_i–variable (the ith coordinate of ∇) so that

$$\begin{aligned}
&\nabla^i \mathbf{F}(u,\mathbf{x}) := \mathbf{F}_u(u,\mathbf{x})\nabla^i u + \mathbf{F}_{x_i}(u,\mathbf{x}),\\
&D_{\mathbf{x}}\cdot\mathbf{F}(\cdot,\mathbf{x}) := F^i_{x_i}(\cdot,\mathbf{x}),\\
&\nabla\cdot\mathbf{F}(u,\mathbf{x}) := F^i_u(u,\mathbf{x})\nabla^i u + D_{\mathbf{x}}\cdot\mathbf{F}(u,\mathbf{x}),
\end{aligned}$$

where we have used the Einstein summation convention that repeated indices are implicitly summed over, which will be used throughout this paper from now on. Second, the Nikolskii space $N^{\kappa,1}, \kappa\in(0,1]$, is defined by

$$v\in N^{\kappa,1} \quad\longleftrightarrow\quad \|v\|_{N^{\kappa,1}} := \mathbb{E}\big[\|v\|_{L^1}\big] + \mathbb{E}\big[|v|_{N^{\kappa,1}}\big] < \infty, \tag{1.4}$$

which forms a Banach space (*cf.* [27] for the deterministic case), where the semi-norm $\mathbb{E}\big[|v|_{N^{\kappa,1}}\big]$ is defined by

$$\mathbb{E}\big[|v|_{N^{\kappa,1}}\big] := \sup_{|\mathbf{h}|>0}\mathbb{E}\Big[\int \frac{|v(\mathbf{y}+\mathbf{h})-v(\mathbf{y})|}{|\mathbf{h}|^{\kappa}}\,\mathrm{d}\mathbf{y}\Big]. \tag{1.5}$$

We assume that the functions involved satisfy the following conditions for $\mathbf{x},\mathbf{y}\in\mathbb{T}^d$:

$$D_{\mathbf{x}}\cdot\mathbf{F}_u(\cdot,\mathbf{x}) = F^j_{ux_j}(\cdot,\mathbf{x})\in L^\infty, \tag{1.6}$$
$$|\mathbf{F}_u(u,\mathbf{x})-\mathbf{F}_u(v,\mathbf{x})| \le C\big(|u|^{p-1}+|v|^{p-1}+1\big)|u-v|^{\kappa_{F1}}, \tag{1.7}$$
$$|D_{\mathbf{x}}\cdot\mathbf{F}(u,\mathbf{x})-D_{\mathbf{y}}\cdot\mathbf{F}(u,\mathbf{y})| \le C\big(|u|^q+1\big)|\mathbf{x}-\mathbf{y}|^{\kappa_{F2}}, \tag{1.8}$$
$$|\sigma(u)-\sigma(v)| \le C|u-v|^{\lambda_\sigma}, \tag{1.9}$$
$$\sup_{i,j}|\boldsymbol{\alpha}_{ij}(u)-\boldsymbol{\alpha}_{ij}(v)| \le C|u-v|^{\gamma_{\boldsymbol{\alpha}}} \tag{1.10}$$

for some constant $C>0$, where $\mathbf{F}_u(u,\cdot)$ and $\mathbf{F}_{x_j}(\cdot,\mathbf{x})$ denote the partial derivatives with respect to u and x_j respectively, $\kappa_{F1}\in(0,1]$, $\kappa_{F2}\in(0,1]$, $\lambda_\sigma\in(\frac{1}{2},1]$, and $\gamma_{\boldsymbol{\alpha}}\in(\frac{1}{2},1]$. We also assume that $D_{\mathbf{x}}\cdot\mathbf{F}(u,\mathbf{x})$ and $\sigma(u)$ have at most linear growth in u, and $\mathbf{A}(u)$ has polynomial growth in u.

The results established in this paper on $\mathbb{T}^d$ can directly be extended to the whole space $\mathbb{R}^d$ by the techniques developed here. For this purpose, it requires to modify the test function in the proof arguments by multiplying a non-negative smooth weight function with appropriate decay rate at infinity. The results established here can also be extended to more general stochastic forcing such as a multidimensional or a cylindrical Brownian motion:

$$\mathrm{d}B = \mathrm{d}B(u,t) = \sum_{k=0}^{m} \langle \Phi(u), \mathrm{d}W_k(t)\mathbf{e}_k\rangle_{\mathcal{H}},$$

where $\mathcal{H}$ is an m-dimensional Hilbert space (with m possibly infinite) with a complete orthonormal basis $\{\mathbf{e}_k\}$, W_k are the independent standard Brownian motions, and $\Phi : \mathbb{R} \to \mathcal{H}$ with $\langle \Phi(u), \mathbf{e}_k\rangle_{\mathcal{H}} = g_k(u)$ and $\sum_k g_k^2(u) \le C(|u|^2+1)$. The results can also be adapted to the additive noise:

$$\mathrm{d}B(\mathbf{x},t) = \sum_{k=0}^{\infty} g_k(\mathbf{x})\mathrm{d}W_k(t),$$

where $\sum_k g_k^2 \in L^1(\mathbb{T}^d)$. It would be interesting to extend our analysis to the noises with all three arguments of form: $B(u,\mathbf{x},t) = \sum_{k=0}^{\infty} g_k(u,\mathbf{x})\mathrm{d}W_k(t)$. There are new difficulties when the noises depend on both solution u and the spatial variable $\mathbf{x}$ in doubling spatial variables in order to quantify the continuous dependence. Essentially, one necessarily comes across the terms where the continuity of g in the two arguments are in competition. This competition manifests itself in the expressions such as $|g(\zeta,\mathbf{y}) - g(\xi,\mathbf{x})|$ under an appropriate integral (see (3.28) and (3.45) below) so that, if $\zeta \to \xi$ first, some terms become unbounded and, if $\mathbf{y} \to \mathbf{x}$ first, the other terms become unbounded.

2. Stochastic Kinetic Formulation

In this section, we introduce the notion of stochastic kinetic solutions for (1.1), motivated by the earlier work in Chen–Perthame [9]; see also Lions–Perthame–Tadmor [25] for the hyperbolic case, and Debussche–Hofmanova–Vovelle [13] and Gess–Souganidis [20] for the translation-invariant degenerate parabolic treatment. Because of the heterogeneity of the flux function $\mathbf{F} = \mathbf{F}(u,\mathbf{x})$, the definition of a stochastic kinetic solution has to be generalized to preserve a structure of *divergence form*; see Definition 2.1 below.

We now motivate the notion of *stochastic kinetic solutions* heuristically as a form of weak solutions. Denote the Heaviside function $H(r) = \mathbb{1}_{r>0}(r)$. Starting from the smooth approximate solutions u^ϵ satisfying the following equation with viscosity:

$$\partial_t u^\epsilon + \nabla\cdot\mathbf{F}(u^\epsilon,\mathbf{x}) = \nabla\cdot\big(\mathbf{A}(u^\epsilon)\nabla u^\epsilon\big) + \sigma(u^\epsilon)\dot{W} + \epsilon\Delta u^\epsilon, \qquad (2.1)$$

we multiply both sides of (2.1) by $-H'(\xi - u^\epsilon)$ for the approximate solution u^ϵ

to obtain

$$
\begin{aligned}
&\partial_t H(\xi-u^\epsilon)\\
&= H'(\xi-u^\epsilon)\mathbf{F}_u(u^\epsilon,\mathbf{x})\cdot\nabla u^\epsilon + H'(\xi-u^\epsilon)D_{\mathbf{x}}\cdot\mathbf{F}(u^\epsilon,\mathbf{x})\\
&\quad -\nabla\cdot\big(H'(\xi-u^\epsilon)\mathbf{A}(u^\epsilon)\nabla u^\epsilon\big)+\mathbf{A}(u^\epsilon):\big(\nabla H'(\xi-u^\epsilon)\otimes\nabla u^\epsilon\big)\\
&\quad - H'(\xi-u^\epsilon)\sigma(u^\epsilon)\dot{W}+\frac{1}{2}H''(\xi-u^\epsilon)\sigma^2(u^\epsilon)\\
&\quad +\epsilon\Delta H(\xi-u^\epsilon)+\epsilon\nabla H'(\xi-u^\epsilon)\cdot\nabla u^\epsilon\\
&= -\nabla\cdot\big(\mathbf{F}_u(\xi,\mathbf{x})H(\xi-u^\epsilon)\big)+\partial_\xi\big(H(\xi-u^\epsilon)D_{\mathbf{x}}\cdot\mathbf{F}(\xi,\mathbf{x})\big)\\
&\quad +\mathbf{A}(\xi):\nabla^2 H(\xi-u^\epsilon)-\delta(\xi-u^\epsilon)\sigma(\xi)\dot{W}\\
&\quad -\partial_\xi\big(\epsilon\delta(\xi-u^\epsilon)|\nabla u^\epsilon|^2+\delta(\xi-u^\epsilon)\mathbf{A}(\xi):(\nabla u^\epsilon\otimes\nabla u^\epsilon)-\frac{1}{2}\delta(\xi-u^\epsilon)\sigma^2(\xi)\big)\\
&\quad +\epsilon\Delta H(\xi-u^\epsilon),
\end{aligned}
\tag{2.2}
$$

where we have used $H'(\xi-u^\epsilon)=\delta(\xi-u^\epsilon)$ and the colon to denote the element-wise scalar product so that $\mathbf{A}:\mathbf{B}=\sum_{1\le i,j\le d}a_{ij}b_{ij}$ for $d\times d$ matrices $\mathbf{A}=(a_{ij})$ and $\mathbf{B}=(b_{ij})$.

Assume that $u^\epsilon(\mathbf{x},t)\to u(\mathbf{x},t)$ *a.e.* as $\epsilon\to 0$. Then, letting $\epsilon\to 0$, we arrive at the *kinetic formulation* of the equation:

$$
\begin{aligned}
&\partial_t H(\xi-u)+\nabla\cdot\big(\mathbf{F}_u(\xi,\mathbf{x})H(\xi-u)\big)-\partial_\xi\big(H(\xi-u)D_{\mathbf{x}}\cdot\mathbf{F}(\xi,\mathbf{x})\big)\\
&\quad =\mathbf{A}(\xi):\nabla^2 H(\xi-u)-\partial_\xi H(\xi-u)\sigma(\xi)\dot{W}-\partial_\xi(m^u+n^u-p^u),
\end{aligned}
\tag{2.3}
$$

where measures m^u, n^u, and $p^u=\sigma(\xi)\delta(\xi-u)\sigma(\xi)$ are the weak limits of the kinetic dissipation, parabolic defect, and Itô correction measures as $\epsilon\to 0$, respectively:

$$
\begin{aligned}
&\epsilon|\nabla u^\epsilon|^2\delta(\xi-u^\epsilon)\rightharpoonup m^u,\\
&\mathbf{A}(\xi):(\nabla u^\epsilon\otimes\nabla u^\epsilon)\,\delta(\xi-u^\epsilon)\rightharpoonup n^u,\\
&\frac{1}{2}\sigma^2(\xi)\delta(\xi-u^\epsilon)\rightharpoonup p^u.
\end{aligned}
$$

Denote by $\mathfrak{M}_1(\mathbb{R})$ the set of probability measures on $\mathbb{R}$ and by $\mathfrak{M}_b^+$ the set of non-negative bounded Radon measures. Moreover, denote C_c^∞ the space of compactly supported smooth functions. Let $\mathcal{L}_\mathbb{R}$ and $\mathcal{L}_{\mathbb{T}^d}$ respectively be the Lebesgue measure on $\mathbb{R}$ and on the flat torus $\mathbb{T}^d$. Let $\mathcal{B}([0,T])$ be the Borel algebra on $[0,T]$ and let $\mathcal{B}(\mathbb{T}^d)$ be the Borel algebra on $\mathbb{T}^d$. Let $\mathcal{P}_T$ be the predictable σ-algebra of $\mathcal{B}([0,T])\otimes\mathcal{F}$; that is, $\mathcal{P}_T$ is generated by all real-valued left-continuous processes adapted to filtration $\{\mathcal{F}_t\}_{t\ge 0}$. The predictable subspace $L^p_P(\Omega\times[0,T]\times\mathbb{T}^d)$ denotes the subspace of functions $\mathbb{P}\otimes\mathcal{L}_\mathbb{R}\otimes\mathcal{L}_{\mathbb{T}^d}$– almost everywhere equal to a $\mathcal{P}_T\otimes\mathcal{B}(\mathbb{T}^d)$-measurable function of $L^p(\Omega\times[0,T]\times\mathbb{T}^d)$ (also see [15, §2.1.1] and the references cited there).

We can now make the following definition, clarifying the roles of the measures exhibited above.

Definition 2.1 (Stochastic kinetic solutions). A function

$$u \in L^p_P(\Omega \times [0,T] \times \mathbb{T}^d) \cap L^p(\Omega; L^\infty([0,T]; L^p(\mathbb{T}^d)))$$

is called a *stochastic kinetic solution* of (1.1)–(1.2) in $\Omega \times \mathbb{T}^d \times [0,T]$ for some $T > 0$ provided that u satisfies the following conditions:

(i) $\nabla \cdot \int_0^u \boldsymbol{\alpha}(\xi)\, \mathrm{d}\xi \in L^2(\Omega \times \mathbb{T}^d \times [0,T])$;

(ii) For any $\varphi \in C_b(\mathbb{R})$ (bounded continuous functions), the Chen–Perthame chain rule relation holds (see [9]):

$$\nabla \cdot \big(\int_0^u \boldsymbol{\alpha}(\xi)\varphi(\xi)\, \mathrm{d}\xi \big) = \varphi(u)\, \nabla \cdot \big(\int_0^u \boldsymbol{\alpha}(\xi)\, \mathrm{d}\xi \big) \tag{2.4}$$

in the sense of distributions in $\mathbb{T}^d$ and almost everywhere in (ω, t);

(iii) For any $\varphi \in C^1_c(\mathbb{R} \times \mathbb{T}^d)$, $t \mapsto \iint H(\xi - u(\mathbf{x},t))\varphi(\xi,\mathbf{x})\, \mathrm{d}\xi\, \mathrm{d}\mathbf{x}$ is càdlàg (*i.e.*, right-continuous with left limits);

(iv) There are non-negative $\mathfrak{M}^+_b(\mathbb{R} \times \mathbb{T}^d \times [0,T])$–valued variables m^u, n^u, and p^u such that

$$\begin{aligned}
&\int_0^T \iint H(\xi - u)\partial_t\varphi\, \mathrm{d}\xi\, \mathrm{d}\mathbf{x}\, \mathrm{d}t + \int_0^T \iint \mathbf{F}_u(\xi,\mathbf{x}) \cdot \nabla\varphi\, \mathrm{d}\xi\, \mathrm{d}\mathbf{x}\, \mathrm{d}t \\
&\quad - \int_0^T \iint D_{\mathbf{x}} \cdot \mathbf{F}(\xi,\mathbf{x}) H(\xi - u)\, \varphi_\xi\, \mathrm{d}\xi\, \mathrm{d}\mathbf{x}\, \mathrm{d}t \\
&= - \int_0^T \iint H(\xi - u)\mathbf{A}(\xi) : \nabla^2\varphi\, \mathrm{d}\xi\, \mathrm{d}\mathbf{x}\, \mathrm{d}t \\
&\quad - \int_0^T \iint \varphi_\xi\, \mathrm{d}(m^u + n^u - p^u)(\xi,\mathbf{x},t) \\
&\quad - \int_0^T \int \sigma(u)\varphi(u,\mathbf{x},t)\, \mathrm{d}\mathbf{x}\, \mathrm{d}W(t) \; + \iint H(\xi - u_0)\varphi(\xi,\mathbf{x},0)\, \mathrm{d}\xi\, \mathrm{d}\mathbf{x}
\end{aligned} \tag{2.5}$$

almost surely, for any $\varphi \in C^\infty_c(\mathbb{R}, \mathbb{T}^d \times [0,T))$. Here, $p^u : \Omega \to \mathfrak{M}^+_b(\mathbb{R} \times \mathbb{T}^d \times \mathbb{R}_+)$ is an *Itô correction measure*:

$$p^u(\varphi) := \frac{1}{2}\int_0^\infty \int_{\mathbb{T}^d} \sigma^2(u)\varphi(u,\mathbf{x},t)\, \mathrm{d}\mathbf{x}\, \mathrm{d}t \quad \text{for any } \varphi \in C_c(\mathbb{R} \times \mathbb{T}^d \times \mathbb{R}_+), \tag{2.6}$$

$n^u : \Omega \to \mathfrak{M}^+_b(\mathbb{R} \times \mathbb{T}^d \times \mathbb{R}_+)$ is the a *parabolic defect measure*:

$$n^u(\varphi) := \int_0^\infty \int_{\mathbb{T}^d} \Big|\nabla \cdot \big(\int_0^{u(\mathbf{x},t)} \boldsymbol{\alpha}(\zeta)\, \mathrm{d}\zeta \big)\Big|^2 \varphi(u(\mathbf{x},t),\mathbf{x},t)\, \mathrm{d}\mathbf{x}\, \mathrm{d}t \tag{2.7}$$

for any $\varphi \in C_c(\mathbb{R}\times\mathbb{T}^d\times\mathbb{R}_+)$, and $m^u : \Omega \to \mathfrak{M}_b^+(\mathbb{R}\times\mathbb{T}^d\times\mathbb{R}_+)$ is the *kinetic defect measure* satisfying that, for any $\varphi \in C_c(\mathbb{R}\times\mathbb{T}^d)$ and $t \in (0,T]$,

$$\int_{\mathbb{R}\times\mathbb{T}^d\times[0,t]} \varphi(\xi,\mathbf{x})\,\mathrm{d}_{(\xi,\mathbf{x},s)}m^u(\xi,\mathbf{x},s;\omega) \in L^2(\Omega\times[0,T])$$

has predictable representative (that is $\mathbb{P}\otimes\mathcal{L}_{\mathbb{R}}$-almost everywhere equal to a function in $L^2_P(\Omega\times[0,T])$).

Remark 2.1. In this section, we introduce the kinetic formulation (2.3) for stochastic kinetic solutions in the sense of (2.5) with the associated kinetic measure m^u, parabolic defect measure n^u, and Itô correction measure p^u in the periodic domain. The existence of stochastic kinetic solutions in the periodic domain will be established in §7. The kinetic formulation can also be defined in $\mathbb{R}^d$ or any other domain, correspondingly. Equation (2.5) is obtained by testing (2.3) with φ and using the Chen–Perthame chain rule (2.4) in [9] (also see [10]). For the isotropic case, the chain rule is not needed (*cf.* [3, 4]).

Remark 2.2. For a stochastic kinetic solution u, we observe that, for any $B_R^c \subset \mathbb{R}$ (the complement of an interval of radius R) and $T \in (0,\infty)$,

$$\lim_{R\to\infty} \mathbb{E}\big[(m^u+n^u)(B_R^c\times\mathbb{T}^d\times[0,T])\big] = 0. \tag{2.8}$$

Remark 2.3. Denote $\bar\nabla := (D_{\mathbf{x}}, -\partial_\xi)$. Then the two integrals involving the flux function $\mathbf{F}$ in (2.5) can be expressed as

$$\int_0^T \iint H(\xi-u)\,(\mathbf{F}_u, D_{\mathbf{x}}\cdot\mathbf{F})\cdot\bar\nabla\varphi\;\mathrm{d}\xi\mathrm{d}\mathbf{x}\mathrm{d}t,$$

which shows clearly the divergence structure attained in this formulation for stochastic kinetic solutions, so that the integral above can be seen as

$$-\int_0^T \iint \bar H(\xi-u)\,(\mathbf{F}_u, D_{\mathbf{x}}\cdot\mathbf{F})\cdot\bar\nabla\varphi\;\mathrm{d}\xi\mathrm{d}\mathbf{x}\mathrm{d}t$$

for $\bar H := 1-H$.

3. Framework for Continuous Dependence Estimates

In this section, we develop a general framework for the continuous dependence estimates of stochastic kinetic solutions. Consider the pair of nonlinear equations:

$$\partial_t u - \nabla\cdot(\mathbf{A}(u)\nabla u) + \nabla\cdot\mathbf{F}(u,\mathbf{x}) = \sigma(u)\dot W, \tag{3.1}$$

$$\partial_t v - \nabla\cdot(\mathbf{B}(v)\nabla v) + \nabla\cdot\mathbf{G}(v,\mathbf{x}) = \tau(v)\dot W, \tag{3.2}$$

where $\mathbf{B}$ is also a positive semi-definite matrix with square root $\boldsymbol{\beta} = (\beta_{ij})$.

Corresponding to assumptions (1.6)–(1.10) for (3.1), we assume the following conditions for (3.2) for $\mathbf{x}, \mathbf{y} \in \mathbb{T}^d$:

$$D_{\mathbf{x}} \cdot \mathbf{G}_u(\cdot, \mathbf{x}) \in L^\infty, \tag{3.3}$$

$$|\mathbf{G}_u(u, \mathbf{x}) - \mathbf{G}_u(v, \mathbf{x})| \le C\big(|u|^{p-1} + |v|^{p-1} + 1\big)|u - v|^{\kappa_{G1}}, \tag{3.4}$$

$$|D_{\mathbf{x}} \cdot \mathbf{G}(u, \mathbf{x}) - D_{\mathbf{y}} \cdot \mathbf{G}(u, \mathbf{y})| \le C\big(|u|^q + 1\big)|\mathbf{x} - \mathbf{y}|^{\kappa_{G2}}, \tag{3.5}$$

$$|\tau(u) - \tau(v)| \le C|u - v|^{\lambda_\tau}, \tag{3.6}$$

$$\sup_{i,j} |\beta_{ij}(u) - \beta_{ij}(v)| \le C|u - v|^{\gamma_{\boldsymbol{\beta}}}. \tag{3.7}$$

We allow κ_{G1}, κ_{G2}, λ_τ, and $\gamma_{\boldsymbol{\beta}}$ to be different from κ_{F1}, κ_{F2}, λ_σ, and $\gamma_{\boldsymbol{\alpha}}$, respectively, but we still assume that $D_{\mathbf{x}} \cdot \mathbf{G}(u, \mathbf{x})$ and $\tau(u)$ have at most linear growth in u and $\mathbf{B}(u)$ has polynomial growth in u. As before, we require that $\kappa_{G1} \in (0, 1]$, $\kappa_{G2} \in (0, 1]$, $\lambda_\tau \in (\frac{1}{2}, 1]$, and $\gamma_{\boldsymbol{\beta}} \in (\frac{1}{2}, 1]$.

We employ the Kruzhkov doubling-of-variable technique and attempt to bound the difference of their stochastic kinetic solutions, so that the stochastic kinetic solution u of (3.1) is understood to take the spatial variable $\mathbf{x}$, and the stochastic kinetic solution v of (3.2) is understood to take the spatial variable $\mathbf{y}$.

In the following, we always assume

$$u_0, v_0 \in L^p(\Omega, \mathcal{P}, \mathrm{d}\mathbb{P}; L^p(\mathbb{T}^d)) \cap L^p(\Omega, \mathcal{P}, \mathrm{d}\mathbb{P}; N^{\kappa,1}(\mathbb{T}^d)).$$

The role of the kinetic function is based on the observation:

$$\int_{\mathbb{R}} H(\xi - u(\mathbf{x}, t))(1 - H(\xi - v(\mathbf{y}, t)))\, \mathrm{d}\xi = (v(\mathbf{y}, t) - u(\mathbf{x}, t))_+.$$

The manipulations are formally only as they stand directly. Thus, we have to make mollifications for justification.

Let $\eta_1 : \mathbb{R} \to \mathbb{R}$ be defined as a smooth convex function, equal to $(\cdot)_+$ outside $[-1, 1] \subseteq \mathbb{R}$, and symmetric with respect to the origin in the sense that $\eta_1'(-r) = 1 - \eta_1'(r)$. Such a function η_1 can be constructed so that $\eta_1'(r) := \int_{-\infty}^{r} \tilde{J}_1(s)\, \mathrm{d}s$, where $\tilde{J}_1$ is a standard symmetric bump function supported on $[-1, 1]$ such as $\tilde{J}_1(r) = C \exp(\frac{1}{1-r^2})$ with choice of C as the normalization constant so that $\int_{\mathbb{R}} \tilde{J}_1(r)\, \mathrm{d}r = 1$. Now scaling by ρ in the usual way to obtain an approximation to $\delta(r)$: $\eta_\rho''(r) = \frac{1}{\rho}\eta_1''(\frac{r}{\rho})$, so that $\eta_\rho'(r)$ preserves the symmetry:

$$1 - \eta_\rho'(r) = \eta_\rho'(-r). \tag{3.8}$$

Finally, we set

$$\eta_\rho(r) := \int_{-\infty}^{r} \eta_\rho'(s)\,\mathrm{d}s. \tag{3.9}$$

By the symmetry, we see that η_ρ coincides with $(\cdot)_+$ outside $[-\rho,\rho]$.

Using the definition of the Heaviside function H and writing $\bar{H}(\zeta - v) := 1 - H(\zeta - v)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} H(\xi - u(\mathbf{x},t))\bar{H}(\zeta - v(\mathbf{y},t))\,\eta_\rho''(\zeta-\xi)\,\mathrm{d}\zeta\mathrm{d}\xi \\
&= \int_{u}^{\infty} \eta_\rho'(v(\mathbf{y},t)-\xi)\,\mathrm{d}\xi = \eta_\rho(v(\mathbf{y},t)-u(\mathbf{x},t)),
\end{aligned} \tag{3.10}$$

while $\eta_\rho(v-u)$ approximates $(v(\mathbf{y},t)-u(\mathbf{x},t))_+$ as $\rho \to 0$.

Define

$$J_\theta(\cdot) := \frac{1}{\theta^d} J(\frac{\cdot}{\theta}), \tag{3.11}$$

where $J : \mathbb{T}^d \to \mathbb{R}$ is a smooth symmetric Friedrichs mollifier on $\mathbb{T}^d$. Then we can further multiply (3.10) by $J_\theta(\mathbf{y}-\mathbf{x})$ and integrate in $\mathrm{d}\mathbf{y}$ to approximate $(v(\mathbf{x},t)-u(\mathbf{x},t))_+$ as $\theta \to 0$.

Before proceeding to the manipulations that mould the equation into a form similar to the terms in (3.10) above, we state a lemma that provides a way to leverage definition (2.5) into a more versatile form. This is essentially Proposition 10 of Debussche–Vovelle [14] (see also [15, Proposition 2.10], [13, Proposition 3.1], and [22, Proposition 3.1]).

Lemma 3.1. *Let u be a stochastic kinetic solution of* (1.1)*. Then there exist representatives $f^\pm(\xi,\mathbf{x};t)$ of $H(\xi-u(\mathbf{x},t)) = \mathbb{1}_{\xi>u(\mathbf{x},t)}(\xi)$ that are almost surely left- and right-continuous-in-time. That is, for any $\psi \in C_c^2(\mathbb{R}\times\mathbb{T}^d)$,*

(i) *for every $\tau \in (0,T]$,*

$$\iint H(\xi-u(\mathbf{x},\tau\pm\varepsilon))\psi(\xi,\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\xi \overset{\varepsilon\to 0}{\longrightarrow} \iint f^\pm(\xi,\mathbf{x},\tau)\psi(\xi,\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\xi \quad a.s.;$$

(ii) *for $\tau = 0$,*

$$\iint H(\xi-u(\mathbf{x},\varepsilon))\psi(\xi,\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\xi \overset{\varepsilon\to 0}{\longrightarrow} \iint f^+(\xi,\mathbf{x};0)\psi(\xi,\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\xi \quad a.s..$$

Moreover, for any $t \in [0,T]$ and $\psi \in C_{\mathrm{c}}^1(\mathbb{R}\times\mathbb{T}^d)$,

$$\iint (f^+ - f^-)(\xi,\mathbf{x};t)\psi(\xi,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\xi = -\int_0^T\iint \mathbb{1}_{\{t\}}(s)\partial_\xi\psi(\xi,\mathbf{x})\,\mathrm{d}m^u(\xi,\mathbf{x},s), \tag{3.12}$$

so that $f^+ = f^- = H(\cdot - u)$ except on at most a countable subset of $[0,T]$.

To arrive at (3.12), we have used the following property of the parabolic measure $n^u(\xi, \mathbf{x}, t)$: For any $t \in [0, T]$ and $\phi \in C_c(\mathbb{R} \times \mathbb{T}^d)$,

$$\begin{aligned}
&\int_0^T \iint \mathbb{1}_{\{t\}}(s)\phi(\xi, \mathbf{x})\,\mathrm{d}n^u(\xi, \mathbf{x}, s) \\
&= \int_0^T \iint \mathbb{1}_{\{t\}}(s)\phi(\xi, \mathbf{x})\Big|\nabla_{\mathbf{x}} \cdot \int_0^{u(\mathbf{x},s)} \boldsymbol{\alpha}(\zeta)\,\mathrm{d}\zeta\Big|^2\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s = 0,
\end{aligned} \tag{3.13}$$

since $\nabla_{\mathbf{x}} \cdot \int_0^{u(\mathbf{x},s)} \boldsymbol{\alpha}(\zeta)\,\mathrm{d}\zeta \in L^2(\Omega \times \mathbb{T}^d \times [0, T])$.

Using the definition of stochastic kinetic solutions in (2.5), we can manipulate to obtain the bounds for the terms in (3.10) above in the following way:

We first derive a version of (2.5) without the temporal integral by choosing a test function of form: $\varphi(\xi, \mathbf{x}, s) = \phi(\xi, \mathbf{x})\chi^\varepsilon(s)$ with

$$\chi^\varepsilon(s) := \begin{cases} 1 & \text{for } s \le t, \\ 1 - \frac{s-t}{\varepsilon} & \text{for } t \le s \le t + \varepsilon, \\ 0 & \text{for } s \ge t + \varepsilon, \end{cases} \tag{3.14}$$

so that $-\partial_s \chi^\varepsilon$ approximates $\delta_t(s)$ as $\varepsilon \to 0$.

Then, from (2.5),

$$\begin{aligned}
&\int_0^T \iint H(\xi - u)\partial_s\left(\phi\,\chi^\varepsilon\right)\,\mathrm{d}\xi\,\mathrm{d}\mathbf{x}\,\mathrm{d}s + \int_0^T \iint H(\xi - u)\,\mathbf{F}_u(\xi, \mathbf{x}) \cdot \nabla\phi\,\chi^\varepsilon\,\mathrm{d}\xi\,\mathrm{d}\mathbf{x}\,\mathrm{d}s \\
&\quad - \int_0^T \iint H(\xi - u)\,D_{\mathbf{x}} \cdot \mathbf{F}(\xi, \mathbf{x})\,\phi_\xi\,\chi^\varepsilon\,\mathrm{d}\xi\,\mathrm{d}\mathbf{x}\,\mathrm{d}s \\
&= -\int_0^T \iint H(\xi - u)\mathbf{A}(\xi) : \nabla^2\phi\,\chi^\varepsilon\,\mathrm{d}\xi\,\mathrm{d}\mathbf{x}\,\mathrm{d}s - \int_0^T \iint \phi_\xi\,\chi^\varepsilon\,\mathrm{d}(m^u + n^u)(\xi, \mathbf{x}, s) \\
&\quad + \frac{1}{2}\int_0^T \int \sigma^2(u)\phi_u(u, \mathbf{x})\,\chi^\varepsilon(s)\,\mathrm{d}\mathbf{x}\,\mathrm{d}s + \int_0^T \int \sigma(u)\phi(u, \mathbf{x})\,\chi^\varepsilon(s)\,\mathrm{d}\mathbf{x}\mathrm{d}W(s) \\
&\quad + \iint H(\xi - u_0)\,\phi(\xi, \mathbf{x})\,\mathrm{d}\xi\,\mathrm{d}\mathbf{x}.
\end{aligned} \tag{3.15}$$

Taking limit $\varepsilon \to 0$ in both sides of (3.15), we apply Lemma 3.1 to obtain

$$H_u^+(\phi) := \iint f^+(\xi, \mathbf{x}, t)\phi\,\mathrm{d}\mathbf{x}\mathrm{d}\xi = I_0^u(\phi) + I_1^u(\phi) + B^u(\phi), \tag{3.16}$$

where $f(\xi, \mathbf{x}, t)$ agrees with $H(\xi - u(\mathbf{x}, t))$ almost surely, except possibly on a countable subset of $[0,T]$, and

$$I_0^u(\phi) = \iint f^+(\xi,\mathbf{x},0)\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}\xi, \tag{3.17}$$

$$\begin{aligned} I_1^u(\phi) = &\int_0^t \iint H(\xi-u)\,\mathbf{F}_u(\xi,\mathbf{x})\cdot\nabla_{\mathbf{x}}\,\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}\xi\,\mathrm{d}s \\ &- \int_0^t \iint D_{\mathbf{x}}\cdot\mathbf{F}(\xi,\mathbf{x})H(\xi-u)\,\phi_\xi \,\mathrm{d}\mathbf{x}\,\mathrm{d}\xi\,\mathrm{d}s \\ &+ \int_0^t \iint H(\xi-u)\,\mathbf{A}(\xi):\nabla_{\mathbf{x}}^2\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}\xi\,\mathrm{d}s \\ &- \frac{1}{2}\int_0^t \int \sigma^2(u(\mathbf{x},s))\phi_u(u(\mathbf{x},s),\mathbf{x}) \,\mathrm{d}\mathbf{x}\,\mathrm{d}s \\ &+ \int_0^t \iint \phi_\xi(\xi,\mathbf{x}) \,\mathrm{d}n^u(\xi,\mathbf{x},s) + \int_0^{t+0} \iint \phi_\xi(\xi,\mathbf{x}) \,\mathrm{d}m^u(\xi,\mathbf{x},s), \end{aligned} \tag{3.18}$$

$$B^u(\phi) = -\int_0^t \int \sigma(u(\mathbf{x},s))\,\phi(u(\mathbf{x},s),\mathbf{x}) \,\mathrm{d}\mathbf{x}\,\mathrm{d}W(s). \tag{3.19}$$

If $\chi^\varepsilon(s)$ in (3.14) is replaced by

$$\chi_\varepsilon(s) := \begin{cases} 1 & \text{for } s \le t-\varepsilon, \\ 1-\frac{s-(t-\varepsilon)}{\varepsilon} & \text{for } t-\varepsilon \le s \le t, \\ 0 & \text{for } s \ge t, \end{cases}$$

we obtain a similar identity to (3.16) with f^+ replaced by f^- and the last term in (3.18) replaced by $\int_0^{t-0}\iint \phi_\xi(\xi,\mathbf{x}) \,\mathrm{d}m^u(\xi,\mathbf{x},s)$.

As for the analogous equation for $\bar{H}(\zeta-v) = 1-H(\zeta-v)$, with analogous representative $\bar{g}^+(\zeta,\mathbf{y},t)$, making the requisite changes in (2.3) directly, we obtain

$$\bar{H}_v^+(\tilde{\phi}) := \iint \bar{g}^+(\zeta,\mathbf{y},t)\tilde{\phi} \,\mathrm{d}\mathbf{y}\,\mathrm{d}\zeta = \bar{I}_0^v(\tilde{\phi}) + \bar{I}_1^v(\tilde{\phi}) + \bar{B}^v(\tilde{\phi}), \tag{3.20}$$

where

$$\bar{I}_0^v(\tilde{\phi}) = -\iint \bar{g}^+(\zeta, \mathbf{y}, 0)\tilde{\phi}\, \mathrm{d}\mathbf{y}\, \mathrm{d}\zeta, \tag{3.21}$$

$$\begin{aligned}
\bar{I}_1^v(\tilde{\phi}) = &\int_0^t \iint \bar{H}(\zeta - v)\, \mathbf{G}_u(\zeta, \mathbf{y}) \cdot \nabla_{\mathbf{y}}\tilde{\phi}\, \mathrm{d}\mathbf{y}\, \mathrm{d}\zeta\, \mathrm{d}s \\
&- \int_0^t \iint D_{\mathbf{x}} \cdot \mathbf{G}(\zeta, \mathbf{y})\bar{H}(\zeta - v)\, \tilde{\phi}_\zeta\, \mathrm{d}\mathbf{y}\, \mathrm{d}\zeta\, \mathrm{d}s \\
&+ \int_0^t \iint \bar{H}(\zeta - v)\, \mathbf{B}(\zeta) : \nabla_{\mathbf{y}}^2\tilde{\phi}\, \mathrm{d}\mathbf{y}\, \mathrm{d}\zeta\, \mathrm{d}s \\
&+ \frac{1}{2}\int_0^t \int \tau^2(v(\mathbf{y}, t))\, \tilde{\phi}_v(v(\mathbf{y}, t), \mathbf{y})\, \mathrm{d}\mathbf{y}\, \mathrm{d}s \\
&- \int_0^t \iint \tilde{\phi}_\zeta(\zeta, \mathbf{y})\, \mathrm{d}n^v(\zeta, \mathbf{y}, s) - \int_0^{t+0} \iint \tilde{\phi}_\zeta(\zeta, \mathbf{y})\, \mathrm{d}m^v(\zeta, \mathbf{y}, s),
\end{aligned} \tag{3.22}$$

$$\bar{B}^v(\tilde{\phi}) = \int_0^t \int \tau(v(\mathbf{y}, t))\, \tilde{\phi}(v(\mathbf{y}, t), \mathbf{y})\, \mathrm{d}\mathbf{y}\, \mathrm{d}W(s). \tag{3.23}$$

Then we can find an expression for the left-hand side of (3.10) via (3.16)–(3.23) by choosing the test functions that will be subsequently prescribed.

3.1. Product Estimate

We can now use the expression for the left-hand side of (3.10).

Proposition 3.1. *Let u be a stochastic kinetic solution of* (3.1) *with initial data u_0, and let v be a stochastic kinetic solution of* (3.2) *with initial data v_0. Let the nonlinear functions in* (3.1)–(3.2) *satisfy* (1.6)–(1.10) *and* (3.3)–(3.7). *Then*

$$\mathbb{E}\big[\iint f^+(\xi, \mathbf{x}, t)\bar{g}^+(\xi, \mathbf{x}, t)\, \mathrm{d}\xi\, \mathrm{d}\mathbf{x}\big] \le \mathbb{E}\big[I^0\big] + \mathbb{E}\big[I^a\big] + \mathbb{E}\big[I^F\big] + \mathbb{E}\big[I^\sigma\big] + \mathbb{E}\big[I^\eta\big], \tag{3.24}$$

where

$$I^0 = \int f^+(\xi, \mathbf{x}, 0)\bar{g}^+(\zeta, \mathbf{y}, 0)\varphi(\xi, \zeta, \mathbf{x}, \mathbf{y})\, \mathrm{d}E, \tag{3.25}$$

$$\begin{aligned} I^a = &\int_0^t \int \bar{H}(\zeta - v)H(\xi - u)\big(\mathbf{A}(\xi) : \nabla^2_{\mathbf{x}}\varphi + \mathbf{B}(\zeta) : \nabla^2_{\mathbf{y}}\varphi\big)\, \mathrm{d}E\, \mathrm{d}s \\ &- \int_0^t \iint \int \varphi(\xi, v, \mathbf{x}, \mathbf{y})\, \mathrm{d}n^u(\xi, \mathbf{x}, s)\, \mathrm{d}\mathbf{y} - \int_0^t \iint \int \varphi(u, \zeta, \mathbf{x}, \mathbf{y})\, \mathrm{d}n^v(\zeta, \mathbf{y}, s)\, \mathrm{d}\mathbf{x}, \end{aligned} \tag{3.26}$$

$$\begin{aligned} I^F = &\int_0^t \int \bar{H}(\zeta - v)H(\xi - u)\big(\mathbf{F}_u(\xi, \mathbf{x}) \cdot \nabla_{\mathbf{x}}\varphi + \mathbf{G}_u(\zeta, \mathbf{y}) \cdot \nabla_{\mathbf{y}}\varphi\big)\, \mathrm{d}E\, \mathrm{d}s \\ &- \int_0^t \int \bar{H}(\zeta - v)H(\xi - u)\big(D_{\mathbf{y}} \cdot \mathbf{G}(\zeta, \mathbf{y}) - D_{\mathbf{x}} \cdot \mathbf{F}(\xi, \mathbf{x})\big)\, \varphi_\zeta\, \mathrm{d}E\, \mathrm{d}s, \end{aligned} \tag{3.27}$$

$$I^\sigma = \frac{1}{2}\int_0^t \iint \big(\tau(v(\mathbf{y}, s)) - \sigma(u(\mathbf{x}, s))\big)^2 \varphi(u, v, \mathbf{x}, \mathbf{y})\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y}\, \mathrm{d}s, \tag{3.28}$$

$$I^\eta = \iint f^+(\xi, \mathbf{x}, t)\bar{g}^+(\xi, \mathbf{x}, t)\, \mathrm{d}\xi\, \mathrm{d}\mathbf{x} - \int f^+(\xi, \mathbf{x}, t)\bar{g}^+(\zeta, \mathbf{y}, t)\varphi(\xi, \zeta, \mathbf{x}, \mathbf{y})\, \mathrm{d}E, \tag{3.29}$$

with $\varphi(\xi, \zeta, \mathbf{x}, \mathbf{y}) := \eta''_\rho(\zeta - \xi)J_\theta(\mathbf{y} - \mathbf{x})$ *and* $\mathrm{d}E := \mathrm{d}\xi\, \mathrm{d}\zeta\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y}$.

Proof. We divide the proof into five steps.

1. For simplicity of notation, we write (3.16) and (3.20) as

$$H_u^+ = I_0^u + I_1^u + B^u \tag{3.30}$$

and

$$\bar{H}_v^+ = \bar{I}_0^v + \bar{I}_1^v + \bar{B}^v, \tag{3.31}$$

by dropping the dependence ϕ and $\tilde{\phi}$ in these functionals when no confusion arises.

Multiplying (3.30) by (3.31), we have

$$\begin{aligned} H_u^+ \bar{H}_v^+ &= \int f^+(\xi, \mathbf{x}, t)\bar{g}^+(\zeta, \mathbf{y}, t)\phi(\xi, \mathbf{x})\tilde{\phi}(\zeta, \mathbf{y})\, \mathrm{d}E \\ &= \int f^+(\xi, \mathbf{x}, t)\bar{g}^+(\zeta, \mathbf{y}, t)\varphi(\xi, \zeta, \mathbf{x}, \mathbf{y})\, \mathrm{d}E \\ &= I_0^u \bar{I}_0^v + I_1^u \bar{H}_v^+ + \bar{I}_1^v H_u^+ - I_1^u \bar{I}_1^v + B^u \bar{B}^v + I_0^u \bar{B}^v + \bar{I}_0^v B^u, \end{aligned} \tag{3.32}$$

where we have denoted $\varphi(\xi, \zeta, \mathbf{x}, \mathbf{y}) = \phi(\xi, \mathbf{x})\tilde{\phi}(\zeta, \mathbf{y})$.

2. Since I_1^u and $\bar{I}_1^v$ are the processes of finite variation (*cf.* [26, Proposi-

tion 0.4.5]), then integrating by parts yields

$$\begin{aligned}
I_1^u \bar{H}_v^+ &= \int_0^t \bar{H}_v^+(s)\,\mathrm{d}I_1^u(s) + \int_0^t I_1^u(s)\,\mathrm{d}\bar{H}_v^+(s) \\
&= \int_0^t \bar{H}_v^+(s-)\,\mathrm{d}I_1^u(s) + \int_0^t I_1^u(s-)\,\mathrm{d}\bar{H}_v^+(s) + \sum \Delta I_1^u(s)\Delta\bar{H}_v^+(s) \\
&= \int_0^t \bar{H}_v^+(s-)\,\mathrm{d}I_1^u(s) + \left(\int_0^t I_1^u(s-)\,\mathrm{d}\bar{I}_1^v(s) + \int_0^t I_1^u(s-)\,\mathrm{d}\bar{B}^v(s)\right) \\
&\quad + \sum \Delta I_1^u(s)\Delta\bar{H}_v^+(s),
\end{aligned}$$

where the sum is over the countable number of points at which the jumps are non-zero. Similarly, we have

$$\begin{aligned}
\bar{I}_1^v H_u^+ &= \int_0^t H_u^+(s)\,\mathrm{d}\bar{I}_1^v(s) + \int_0^t \bar{I}_1^v(s)\,\mathrm{d}H_u^+(s) \\
&= \int_0^t H_u^+(s-)\,\mathrm{d}\bar{I}_1^v(s) + \left(\int_0^t \bar{I}_1^v(s-)\,\mathrm{d}I_1^u(s) + \int_0^t \bar{I}_1^v(s-)\,\mathrm{d}B^u(s)\right) \\
&\quad + \sum \Delta\bar{I}_1^v(s)\Delta H_u^+(s).
\end{aligned}$$

Furthermore, we obtain

$$I_1^u \bar{I}_1^v = \int_0^t I_1^u(s-)\,\mathrm{d}\bar{I}_1^v(s) + \int_0^t \bar{I}_1^v(s-)\,\mathrm{d}I_1^u(s) + \sum \Delta I_1^u(s)\Delta\bar{I}_1^v(s). \tag{3.33}$$

The only jumps that may occur come from the terms, $m^u(\phi_\xi \times [0,s])$ in I_1^u and $-m^v(\tilde{\phi}_\zeta \times [0,s])$ in $\bar{I}_1^v$, so that

$$\begin{aligned}
\Delta H_u^+(s) &= \Delta I_1^u(s) = \Delta m^u(\phi_\xi \times [0,s]), \\
\Delta\bar{H}_v^+(s) &= \Delta\bar{I}_1^v(s) = \Delta m^v(\tilde{\phi}_\zeta \times [0,s]).
\end{aligned} \tag{3.34}$$

Therefore, we have

$$\begin{aligned}
&I_1^u \bar{H}_v^+ + \bar{I}_1^v H_u^+ - I_1^u \bar{I}_1^v \\
&= \int_0^t \bar{H}_v^+(s-)\,\mathrm{d}I_1^u(s) + \int_0^t H_u^+(s-)\,\mathrm{d}\bar{I}_1^v(s) + \int_0^t I_1^u(s-)\,\mathrm{d}\bar{B}^v(s) \\
&\quad + \int_0^t \bar{I}_1^v(s-)\,\mathrm{d}B^u(s) + \sum \Delta\bar{H}_v^+(s)\Delta H_u^+(s).
\end{aligned}$$

Next we claim that

$$\begin{aligned}
&\int_0^t \bar{H}_v^+(s-)\,\mathrm{d}I_1^u(s) + \int_0^t H_u^+(s-)\,\mathrm{d}\bar{I}_1^v(s) + \sum \Delta\bar{H}_v^+(s)\Delta H_u^+(s) \\
&= \int_0^t H_u^+(s)\,\mathrm{d}\bar{I}_1^v(s) + \int_0^t \bar{H}_v^+(s)\,\mathrm{d}I_1^u(s).
\end{aligned} \tag{3.35}$$

This can be seen by using (3.34) to obtain

$$\int_0^t \bar{H}_v^+(s)\,\mathrm{d}I_1^u(s) - \int_0^t \bar{H}_v^+(s-)\,\mathrm{d}I_1^u(s) = \int_0^t \bar{I}_1^v(s)\,\mathrm{d}I_1^u(s) - \int_0^t \bar{I}_1^v(s-)\,\mathrm{d}I_1^u(s),$$
$$\int_0^t H_u^+(s)\,\mathrm{d}\bar{I}_1^v(s) - \int_0^t H_u^+(s-)\,\mathrm{d}\bar{I}_1^v(s) = \int_0^t I_1^u(s)\,\mathrm{d}\bar{I}_1^v(s) - \int_0^t I_1^u(s-)\,\mathrm{d}\bar{I}_1^v(s),$$

from which the claim follows by (3.33). With (3.35), we can conclude that

$$\begin{aligned}
&\int f^+(\xi,\mathbf{x},t)\bar{g}^+(\zeta,\mathbf{y},t)\varphi(\xi,\zeta,\mathbf{x},\mathbf{y})\;\mathrm{d}E \\
&= I_0^u\bar{I}_0^v + \int_0^t \iint \bar{H}(\zeta - v(\mathbf{y},s))\tilde{\phi}_\zeta\;\mathrm{d}\zeta\mathrm{d}\mathbf{y}\mathrm{d}I_1^u \\
&\quad + \int_0^t \iint H(\xi - u(\mathbf{x},s))\phi_\xi\;\mathrm{d}\xi\mathrm{d}\mathbf{x}\mathrm{d}\bar{I}_1^v + B^u\bar{B}^v + \mathscr{M},
\end{aligned} \tag{3.36}$$

where $\mathscr{M}$ denotes a martingale term, which has expectation zero.

3. Next we have

$$I_0^u\bar{I}_0^v = \int f^+(\xi,\mathbf{x},0)\bar{g}^+(\zeta,\mathbf{y},0)\varphi(\xi,\zeta,\mathbf{x},\mathbf{y})\;\mathrm{d}E. \tag{3.37}$$

By the Itô isometry,

$$\begin{aligned}
\mathbb{E}\big[B^u\bar{B}^v\big] &= -\mathbb{E}\Big[\int_0^t \int \sigma(u(\mathbf{x},s))\phi(u(\mathbf{x},s),\mathbf{x})\;\mathrm{d}\mathbf{x}\mathrm{d}W(s) \\
&\qquad\quad \times \int_0^t \int \tau(v(\mathbf{y},s))\tilde{\phi}(v(\mathbf{y},s),\mathbf{y})\;\mathrm{d}\mathbf{y}\mathrm{d}W(s)\Big] \\
&= -\mathbb{E}\Big[\int_0^t \iint \sigma(u)\tau(v)\varphi(u,v,\mathbf{x},\mathbf{y})\;\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s\Big].
\end{aligned} \tag{3.38}$$

4. By a density argument via the monotone class theorem (see *e.g.*, [28, §2.3.1] or [14, (24)–(25)] in the context), we can choose

$$\varphi(\xi,\zeta,\mathbf{x},\mathbf{y}) := \eta_\rho''(\zeta-\xi)J_\theta(\mathbf{y}-\mathbf{x}) \geq 0, \tag{3.39}$$

where η_ρ and J_θ are defined as in (3.9) and (3.11). With such a choice of the test function, we have the following usual identities:

$$\nabla_{\mathbf{x}}\varphi + \nabla_{\mathbf{y}}\varphi = 0, \quad \nabla_{\mathbf{x}}^2\varphi - \nabla_{\mathbf{y}}^2\varphi = 0, \quad \varphi_\xi + \varphi_\zeta = 0. \tag{3.40}$$

Combining (3.37)–(3.38) with (3.36) and using $\varphi_\xi = -\varphi_\zeta$, we have

$$
\begin{aligned}
&\mathbb{E}\big[\int f^+(\xi,\mathbf{x},t)\bar{g}^+(\zeta,\mathbf{y},t)\varphi(\xi,\zeta,\mathbf{x},\mathbf{y})\,\mathrm{d}E\big]\\
&=\mathbb{E}\big[I^0\big]+\mathbb{E}\big[I^a\big]+\mathbb{E}\big[I^F\big]+\mathbb{E}\big[I^\sigma\big]\\
&\quad-\int_0^{t+0}\iint\int\varphi(\xi,v^+,\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}m^u(\xi,\mathbf{x},s)\\
&\quad-\int_0^{t+0}\iint\int\varphi(u^+,\zeta,\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}m^v(\zeta,\mathbf{y},s)\\
&\le\mathbb{E}\big[I^0\big]+\mathbb{E}\big[I^a\big]+\mathbb{E}\big[I^F\big]+\mathbb{E}\big[I^\sigma\big],
\end{aligned}
$$

since m^u and m^v are non-negative Radon measures, and $\mathbb{E}\big[\mathscr{M}\big]=0$, where u^+ and v^+ are the right continuous versions of u and v, respectively, and there is no distinction between u and u^+ within a time integral against a non-atomic measure.

5. By the definition of I^η, we conclude (3.24).

3.2. Difference Estimates

From (3.24), we need to estimate the integral terms $\mathbb{E}\big[I^0\big]$, $\mathbb{E}\big[I^a\big]$, $\mathbb{E}\big[I^F\big]$, $\mathbb{E}\big[I^\sigma\big]$, and $\mathbb{E}\big[I^\eta\big]$ as defined in (3.25)–(3.29). We refer to these integral terms as the *initial term*, *parabolic term*, *flux term*, *Itô correction term*, and *mollification term*, respectively.

Proposition 3.2. *Let u be a stochastic kinetic solution of* (3.1) *with initial data u_0, and let v be a stochastic kinetic solution of* (3.2) *with initial data v_0. Let η_ρ and J_θ be defined as in* (3.9) *and* (3.11). *Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ satisfy* (1.10) *with respective indices $\gamma_{\boldsymbol{\alpha}}$ and $\gamma_{\boldsymbol{\beta}}$, and let σ and τ satisfy* (1.9) *with respective indices λ_σ and λ_τ. Assume that*

$$\|\sqrt{\mathbf{B}}-\sqrt{\mathbf{A}}\|_{L^\infty}:=\sup_{i,j}\|\beta_{ij}-\alpha_{ij}\|_{L^\infty}<\infty, \tag{3.41}$$

$$\|(\mathbf{G}_u-\mathbf{F}_u, D_{\mathbf{x}}\cdot(\mathbf{G}-\mathbf{F}),\tau-\sigma)\|_{L^\infty}<\infty. \tag{3.42}$$

Then the following estimates hold:

(i) *For the parabolic term,*

$$
\begin{aligned}
\mathbb{E}\big[I^a\big]\le\,& d\theta^{-2}\big(\|\sqrt{\mathbf{B}}-\sqrt{\mathbf{A}}\|^2_{L^\infty}+C(\boldsymbol{\alpha})\rho^{\gamma_{\boldsymbol{\alpha}}}(\|\sqrt{\mathbf{B}}-\sqrt{\mathbf{A}}\|_{L^\infty}+\rho^{\gamma_{\boldsymbol{\alpha}}})\big)\\
&\times\mathbb{E}\big[\int_0^t\iint\eta_\rho(v(\mathbf{y},s)-u(\mathbf{x},s))J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s\big],
\end{aligned}
\tag{3.43}
$$

where $C(\boldsymbol{\alpha})\ge\|\boldsymbol{\alpha}\|_{C^{\gamma_{\boldsymbol{\alpha}}}}$.

(ii) *For the flux term,*

$$
\begin{aligned}
\mathbb{E}\big[I^F\big] \leq & \, C\|D_{\mathbf{x}} \cdot \mathbf{F}_u\|_{L^\infty}\mathbb{E}\Big[\int_0^t \iint \eta_\rho(v(\mathbf{y},s)-u(\mathbf{x},s))J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s\Big] \\
& + C\big(\theta^{-1}\|\mathbf{G}_u - \mathbf{F}_u\|_{L^\infty} + \rho^{-1}\|D_{\mathbf{x}} \cdot (\mathbf{G}-\mathbf{F})\|_{L^\infty}\big) \\
& \quad \times \mathbb{E}\Big[\int_0^t \iint \eta_\rho(v(\mathbf{y},s)-u(\mathbf{x},s))J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s\Big] \\
& + C\big(\rho^{\kappa_{F1}}\theta^{-1} + \theta^{\kappa_{F2}}\big)\mathbb{E}\Big[\int_0^t \int \big(|(u,v)|^p + |(u,v)|^q + 1\big)\,\mathrm{d}\mathbf{x}\mathrm{d}s\Big],
\end{aligned}
\tag{3.44}
$$

where C *depends on* d, $|\mathbb{T}^d|$, $\mathbf{F}$, *and* $\mathbf{G}$.

(iii) *For the Itô correction term,*

$$
\mathbb{E}[I^\sigma] \leq Ct\rho^{-1}\big(\|\tau-\sigma\|_{L^\infty}^2 + \rho^{2\lambda_\sigma}\big), \tag{3.45}
$$

where C *is a constant depending on* d, $|\mathbb{T}^d|$, σ, *and* τ.

(iv) *For the mollification term,*

$$
\mathbb{E}[I^\eta] = o_{\theta,\rho}(1) \to 0 \qquad \text{as } \theta,\rho \to 0. \tag{3.46}
$$

Proof. We divide the proof into four steps.

1. *Parabolic terms.* With reference to (3.26) where I^a is defined, we first show

$$
\begin{aligned}
I^a \leq & \int_0^t \int \bar{H}(\zeta - v)H(\xi - u)\,(\boldsymbol{\beta}(\zeta)-\boldsymbol{\alpha}(\zeta))\,(\boldsymbol{\beta}(\zeta)-\boldsymbol{\alpha}(\zeta)) : \nabla^2_{\mathbf{xy}}\varphi\,\mathrm{d}E\mathrm{d}s \\
& + \int_0^t \int \bar{H}(\zeta - v)H(\xi - u)\,(\boldsymbol{\beta}(\zeta)-\boldsymbol{\alpha}(\zeta))\,(\boldsymbol{\alpha}(\zeta)-\boldsymbol{\alpha}(\xi)) : \nabla^2_{\mathbf{xy}}\varphi\,\mathrm{d}E\mathrm{d}s \\
& + \int_0^t \int \bar{H}(\zeta - v)H(\xi - u)\,(\boldsymbol{\alpha}(\zeta)-\boldsymbol{\alpha}(\xi))\,(\boldsymbol{\beta}(\zeta)-\boldsymbol{\alpha}(\zeta)) : \nabla^2_{\mathbf{xy}}\varphi\,\mathrm{d}E\mathrm{d}s \\
& + \int_0^t \int \bar{H}(\zeta - v)H(\xi - u)\,(\boldsymbol{\alpha}(\zeta)-\boldsymbol{\alpha}(\xi))\,(\boldsymbol{\alpha}(\zeta)-\boldsymbol{\alpha}(\xi)) : \nabla^2_{\mathbf{xy}}\varphi\,\mathrm{d}E\mathrm{d}s.
\end{aligned}
\tag{3.47}
$$

First, by (3.40), we have

$$\begin{aligned}
&\int_0^t \int \bar{H}(\zeta - v)H(\xi - u)\big(\mathbf{A}(\xi) : \nabla_{\mathbf{x}}^2\varphi + \mathbf{B}(\zeta) : \nabla_{\mathbf{y}}^2\varphi\big) \,\mathrm{d}E\mathrm{d}s \\
&= -\int_0^t \int \bar{H}(\zeta - v)H(\xi - u)\big(\mathbf{A}(\xi) - \boldsymbol{\alpha}(\xi)\boldsymbol{\beta}(\zeta) - \boldsymbol{\beta}(\zeta)\boldsymbol{\alpha}(\xi) + \mathbf{B}(\zeta)\big) : \nabla_{\mathbf{xy}}^2\varphi \,\mathrm{d}E\mathrm{d}s \\
&\quad - \int_0^t \int \bar{H}(\zeta - v)H(\xi - u)\big(\boldsymbol{\alpha}(\xi)\boldsymbol{\beta}(\zeta) + \boldsymbol{\beta}(\zeta)\boldsymbol{\alpha}(\xi)\big) : \nabla_{\mathbf{xy}}^2\varphi \,\mathrm{d}E\mathrm{d}s.
\end{aligned} \tag{3.48}$$

Using the chain rule (2.4) for stochastic kinetic solutions and the symmetry of $\nabla_{\mathbf{xy}}^2\varphi$, we have

$$\begin{aligned}
&\int_0^t \int \bar{H}(\zeta - v)H(\xi - u)\big(\boldsymbol{\alpha}(\xi)\boldsymbol{\beta}(\zeta) + \boldsymbol{\beta}(\zeta)\boldsymbol{\alpha}(\xi)\big) : \nabla_{\mathbf{xy}}^2\varphi \,\mathrm{d}E\mathrm{d}s \\
&= \int_0^t \int \nabla_{\mathbf{x}} H(\xi - u) \otimes \nabla_{\mathbf{y}} \bar{H}(\zeta - v) : \boldsymbol{\alpha}(\xi)\boldsymbol{\beta}(\zeta)\,\varphi \,\mathrm{d}E\mathrm{d}s \\
&\quad + \int_0^t \int \nabla_{\mathbf{y}} \bar{H}(\zeta - v) \otimes \nabla_{\mathbf{x}} H(\xi - u) : \boldsymbol{\alpha}(\xi)\boldsymbol{\beta}(\zeta)\,\varphi \,\mathrm{d}E\mathrm{d}s \\
&= -\int_0^t \iint \nabla_{\mathbf{yx}}^2 : \int_\infty^u \boldsymbol{\alpha}(\xi) \int_{-\infty}^v \boldsymbol{\beta}(\zeta)\eta_\rho''(\zeta - \xi)\,\mathrm{d}\zeta\,\mathrm{d}\xi\; J_\theta(\mathbf{y} - \mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s \\
&\quad - \int_0^t \iint \nabla_{\mathbf{yx}}^2 : \int_\infty^u \int_{-\infty}^v \boldsymbol{\alpha}(\xi)\boldsymbol{\beta}(\zeta)\eta_\rho''(\zeta - \xi)\,\mathrm{d}\zeta\,\mathrm{d}\xi\; J_\theta(\mathbf{y} - \mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s \\
&= -2\int_0^t \iint \nabla_{\mathbf{x}} \otimes \nabla_{\mathbf{y}} : \Big(\int_0^u \boldsymbol{\alpha}(\xi)\,\mathrm{d}\xi\Big)\Big(\int_0^v \boldsymbol{\beta}(\zeta)\,\mathrm{d}\zeta\Big)\varphi(u, v, \mathbf{x}, \mathbf{y})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s,
\end{aligned} \tag{3.49}$$

where we have also used the following fact:

$$\nabla_{\mathbf{x}} \cdot \Big(\int_r^u \boldsymbol{\alpha}(\xi)\eta_\rho''(\zeta - \xi)\,\mathrm{d}\xi\Big) = \nabla_{\mathbf{x}} \cdot \Big(\int_0^u \boldsymbol{\alpha}(\xi)\eta_\rho''(\zeta - \xi)\,\mathrm{d}\xi\Big) \qquad \text{for any fixed } r.$$

Next, we employ form (2.7) of the parabolic defect measure with $\varphi \in C_c^\infty(\mathbb{R}^2 \times (\mathbb{T}^d)^2)$ to obtain

$$\begin{aligned}
&\int_0^t \iint \varphi(\xi, v(\mathbf{y}, s), \mathbf{x}, \mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}n^u(\xi, \mathbf{x}, s) \\
&= \int_0^t \iint \eta_\rho''(v(\mathbf{y}, s) - u(\mathbf{x}, s))J_\theta(\mathbf{y} - \mathbf{x})\Big|\nabla_{\mathbf{x}} \cdot \int_0^{u(\mathbf{x},s)} \boldsymbol{\alpha}(\zeta)\,\mathrm{d}\zeta\Big|^2 \,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s,
\end{aligned}$$

where we have used that $\nabla_{\mathbf{x}} \cdot \int_0^{u(\mathbf{x},s)} \boldsymbol{\alpha}(\zeta)\,\mathrm{d}\zeta \in L^2(\Omega \times \mathbb{T}^d \times [0, T])$.

Similarly, we have

$$
\begin{aligned}
&\int_0^t \iint \int \varphi(u(\mathbf{x},s),\zeta,\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}\mathrm{d}n^v(\zeta,\mathbf{y},s)\\
&= \int_0^t \iint \eta_\rho''(v(\mathbf{y},s)-u(\mathbf{x},s))J_\theta(\mathbf{y}-\mathbf{x})\Big|\nabla_{\mathbf{y}}\cdot\int_0^{v(\mathbf{y},s)}\boldsymbol{\beta}(\xi)\,\mathrm{d}\xi\Big|^2\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s,
\end{aligned}
$$

by using $\nabla_{\mathbf{y}}\cdot\int_0^{v(\mathbf{y},s)}\boldsymbol{\beta}(\xi)\,\mathrm{d}\xi\in L^2(\Omega\times\mathbb{T}^d\times[0,T])$.

Therefore, we obtain

$$
\begin{aligned}
&-\int_0^t\int \bar{H}(\zeta-v)H(\xi-u)\big(\boldsymbol{\alpha}(\xi)\boldsymbol{\beta}(\zeta)+\boldsymbol{\beta}(\zeta)\boldsymbol{\alpha}(\xi)\big):\nabla^2_{\mathbf{xy}}\varphi\,\mathrm{d}E\mathrm{d}s\\
&\quad-\int_0^t\iint\int\varphi(\xi,v,\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\mathrm{d}n^u(\xi,\mathbf{x},s)-\int_0^t\iint\int\varphi(u,\zeta,\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}\mathrm{d}n^v(\zeta,\mathbf{y},s)\\
&=2\int_0^t\nabla^2_{\mathbf{xy}}:\Big(\int_0^u\boldsymbol{\alpha}(\xi)\,\mathrm{d}\xi\Big)\Big(\int_0^v\boldsymbol{\beta}(\zeta)\,\mathrm{d}\zeta\Big)\varphi(u,v,\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s\\
&\quad-\int_0^t\iint\Big(\Big|\nabla_{\mathbf{x}}\cdot\int_0^u\boldsymbol{\alpha}(\zeta)\,\mathrm{d}\zeta\Big|^2+\Big|\nabla_{\mathbf{y}}\cdot\int_0^v\boldsymbol{\beta}(\xi)\,\mathrm{d}\xi\Big|^2\Big)\varphi(u,v,\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s\le 0.
\end{aligned}
$$

Inserting this into (3.48) yields

$$
\begin{aligned}
I^a&\le\int_0^t\int\bar{H}(\zeta-v)H(\xi-u)\big(\mathbf{A}(\xi)-\boldsymbol{\alpha}(\xi)\boldsymbol{\beta}(\zeta)-\boldsymbol{\beta}(\zeta)\boldsymbol{\alpha}(\xi)+\mathbf{B}(\zeta)\big):\nabla^2_{\mathbf{xy}}\varphi\,\mathrm{d}E\mathrm{d}s\\
&=\int_0^t\int\bar{H}(\zeta-v)H(\xi-u)\big(\boldsymbol{\alpha}(\xi)-\boldsymbol{\beta}(\zeta)\big)\big(\boldsymbol{\alpha}(\xi)-\boldsymbol{\beta}(\zeta)\big):\nabla^2_{\mathbf{xy}}\varphi\,\mathrm{d}E\mathrm{d}s.
\end{aligned}
\tag{3.50}
$$

Notice that

$$
\begin{aligned}
&\big(\boldsymbol{\beta}(\zeta)-\boldsymbol{\alpha}(\xi)\big)\big(\boldsymbol{\beta}(\zeta)-\boldsymbol{\alpha}(\xi)\big)\\
&=\left(\boldsymbol{\beta}(\zeta)-\boldsymbol{\alpha}(\zeta)\right)\left(\boldsymbol{\beta}(\zeta)-\boldsymbol{\alpha}(\zeta)\right)+\left(\boldsymbol{\beta}(\zeta)-\boldsymbol{\alpha}(\zeta)\right)\left(\boldsymbol{\alpha}(\zeta)-\boldsymbol{\alpha}(\xi)\right)\\
&\quad+\left(\boldsymbol{\alpha}(\zeta)-\boldsymbol{\alpha}(\xi)\right)\left(\boldsymbol{\beta}(\zeta)-\boldsymbol{\alpha}(\zeta)\right)+\left(\boldsymbol{\alpha}(\xi)-\boldsymbol{\alpha}(\zeta)\right)\left(\boldsymbol{\alpha}(\xi)-\boldsymbol{\alpha}(\zeta)\right).
\end{aligned}
$$

Combining this with (3.50), we complete the proof of (3.47).

These terms in I^a can be estimated by invoking either the boundedness of

$\|\sqrt{\mathbf{B}}-\sqrt{\mathbf{A}}\|_{L^\infty}$ or the continuity of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in (1.10) as follows:

$$\begin{aligned}
&\left|\int_0^t\int \bar{H}(\zeta-v)H(\xi-u)\left(\boldsymbol{\beta}(\zeta)-\boldsymbol{\alpha}(\zeta)\right)\left(\boldsymbol{\beta}(\zeta)-\boldsymbol{\alpha}(\zeta)\right):\nabla^2_{\mathbf{x}}\varphi \,\mathrm{d}E\mathrm{d}s\right|\\
&\le C\int_0^t\int \bar{H}(\zeta-v)H(\xi-u)\|\sqrt{\mathbf{B}}-\sqrt{\mathbf{A}}\|^2_{L^\infty}\theta^{-2}J_\theta(\mathbf{y}-\mathbf{x})\eta''_\rho(\zeta-\xi)\,\mathrm{d}E\mathrm{d}s\\
&\le C\theta^{-2}\|\sqrt{\mathbf{B}}-\sqrt{\mathbf{A}}\|^2_{L^\infty}\int_0^t\iint \eta_\rho(v-u)J_\theta(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s. \qquad (3.51)
\end{aligned}$$

Using $\gamma_{\boldsymbol{\alpha}}$ as the Hölder exponent of $\boldsymbol{\alpha}$, we have the estimates:

$$\begin{aligned}
&\left|\int_0^t\int \bar{H}(\zeta-v)H(\xi-u)\left(\boldsymbol{\alpha}(\zeta)-\boldsymbol{\alpha}(\xi)\right)\left(\boldsymbol{\beta}(\zeta)-\boldsymbol{\alpha}(\zeta)\right):\nabla^2_{\mathbf{x}}\varphi \,\mathrm{d}E\mathrm{d}s\right|\\
&\le d\|\sqrt{\mathbf{B}}-\sqrt{\mathbf{A}}\|_{L^\infty}\\
&\quad\times\int_0^t\int \bar{H}(\zeta-v)H(\xi-u)\frac{|\boldsymbol{\alpha}(\zeta)-\boldsymbol{\alpha}(\xi)|}{|\zeta-\xi|^{\gamma_{\boldsymbol{\alpha}}}}\rho^{\gamma_{\boldsymbol{\alpha}}}|\nabla^2_{\mathbf{x}}J_\theta|\eta''_\rho(\zeta-\xi)\,\mathrm{d}E\mathrm{d}s\\
&\le dC(\boldsymbol{\beta})\rho^{\gamma_{\boldsymbol{\alpha}}}\theta^{-2}\|\sqrt{\mathbf{B}}-\sqrt{\mathbf{A}}\|_{L^\infty}\int_0^t\iint \eta_\rho(v-u)J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s, \qquad (3.52)
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\left|\int_0^t\int \bar{H}(\zeta-v)H(\xi-u)\left(\boldsymbol{\beta}(\zeta)-\boldsymbol{\alpha}(\zeta)\right)\left(\boldsymbol{\alpha}(\zeta)-\boldsymbol{\alpha}(\xi)\right):\nabla^2_{\mathbf{x}}\varphi \,\mathrm{d}E\mathrm{d}s\right|\\
&\le dC(\boldsymbol{\alpha})\rho^{\gamma_{\boldsymbol{\alpha}}}\theta^{-2}\|\sqrt{\mathbf{B}}-\sqrt{\mathbf{A}}\|_{L^\infty}\int_0^t\iint \eta_\rho(v-u)J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s,
\end{aligned}$$

where $C(\boldsymbol{\alpha})\ge\|\boldsymbol{\alpha}\|_{C^{\gamma_{\boldsymbol{\alpha}}}}$, and we have used the estimate: $|\nabla^2_{\mathbf{x}}J_\theta(\mathbf{x})|\le C\theta^{-2}J_\theta(\mathbf{x})$. An analogous estimate also holds: $|\nabla_{\mathbf{x}}J_\theta(\mathbf{x})|\le C\theta^{-1}J_\theta(\mathbf{x})$ — which will be used below. Finally, we have the estimate:

$$\begin{aligned}
&\left|\int_0^t\int \bar{H}(\zeta-v)H(\xi-u)\left(\boldsymbol{\alpha}(\xi)-\boldsymbol{\alpha}(\zeta)\right)\left(\boldsymbol{\alpha}(\xi)-\boldsymbol{\alpha}(\zeta)\right):\nabla^2_{\mathbf{x}}\varphi \,\mathrm{d}E\right|\,\mathrm{d}s\\
&\le dC(\boldsymbol{\alpha})\rho^{2\gamma_{\boldsymbol{\alpha}}}\theta^{-2}\int_0^t\iint \eta_\rho(v-u)J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s. \qquad (3.53)
\end{aligned}$$

Combining all the estimates above to conclude (3.43).

2. *Flux terms.* Notice that

$$
\begin{aligned}
&\mathbf{F}_u(\xi,\mathbf{x})\cdot\nabla_{\mathbf{x}}\varphi+\mathbf{G}_u(\zeta,\mathbf{y})\cdot\nabla_{\mathbf{y}}\varphi\\
&=\big(\mathbf{G}_u(\zeta,\mathbf{y})-\mathbf{F}_u(\zeta,\mathbf{y})\big)\cdot\nabla_{\mathbf{y}}\varphi+\big(\mathbf{F}_u(\zeta,\mathbf{y})-\mathbf{F}_u(\xi,\mathbf{y})\big)\cdot\nabla_{\mathbf{y}}\varphi\\
&\quad+\big(\mathbf{F}_u(\xi,\mathbf{y})-\mathbf{F}_u(\xi,\mathbf{x})\big)\cdot\nabla_{\mathbf{y}}\varphi+\mathbf{F}_u(\xi,\mathbf{x})\cdot\big(\nabla_{\mathbf{x}}\varphi+\nabla_{\mathbf{y}}\varphi\big),
\end{aligned}\tag{3.54}
$$

$$
\begin{aligned}
&D_{\mathbf{y}}\cdot\mathbf{G}(\zeta,\mathbf{y})-D_{\mathbf{x}}\cdot\mathbf{F}(\xi,\mathbf{x})\\
&=\big(D_{\mathbf{y}}\cdot\mathbf{G}(\zeta,\mathbf{y})-D_{\mathbf{y}}\cdot\mathbf{F}(\zeta,\mathbf{y})\big)+\big(D_{\mathbf{y}}\cdot\mathbf{F}(\zeta,\mathbf{y})-D_{\mathbf{x}}\cdot\mathbf{F}(\zeta,\mathbf{x})\big)\\
&\quad+\big(D_{\mathbf{x}}\cdot\mathbf{F}(\zeta,\mathbf{x})-D_{\mathbf{x}}\cdot\mathbf{F}(\xi,\mathbf{x})\big).
\end{aligned}\tag{3.55}
$$

First, with condition (3.42), we have

$$
\begin{aligned}
&\Big|\int_0^t\int\bar{H}(\zeta-v)H(\xi-u)\big(\mathbf{G}_u(\zeta,\mathbf{y})-\mathbf{F}_u(\zeta,\mathbf{y})\big)\cdot\nabla_{\mathbf{y}}\varphi\,\mathrm{d}E\mathrm{d}s\Big|\\
&\le C\theta^{-1}\|\mathbf{G}_u-\mathbf{F}_u\|_{L^\infty}\int_0^t\iint\eta_\rho(v-u)J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s,
\end{aligned}\tag{3.56}
$$

$$
\begin{aligned}
&\Big|\int_0^t\int\bar{H}(\zeta-v)H(\xi-u)\big(D_{\mathbf{x}}\cdot\mathbf{G}(\zeta,\mathbf{y})-D_{\mathbf{x}}\cdot\mathbf{F}(\zeta,\mathbf{y})\big)\,\varphi_\zeta\,\mathrm{d}E\mathrm{d}s\Big|\\
&\le C\rho^{-1}\|D_{\mathbf{x}}\cdot(\mathbf{G}-\mathbf{F})\|_{L^\infty}\int_0^t\iint\eta_\rho(v-u)J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s.
\end{aligned}\tag{3.57}
$$

Next, for $\Gamma(\xi,\zeta,\mathbf{y})=|\mathbf{F}_u(\zeta,\mathbf{y})-\mathbf{F}_u(\xi,\mathbf{y})||\zeta-\xi|^{-\kappa_{F1}}$ with

$$
\Gamma(\xi,\zeta,\mathbf{y})\le C\big(|\xi|^{p-1}+|\zeta|^{p-1}+1\big),
$$

consider the following calculation:

$$
\begin{aligned}
&\iint\bar{H}(\zeta-v)H(\xi-u)\Gamma(\xi,\zeta,\mathbf{y})|\zeta-\xi|^{\kappa_{F1}}\eta_\rho''(\zeta-\xi)\,\mathrm{d}\xi\mathrm{d}\zeta\\
&=\int_u^\infty\int_{-\infty}^v\Gamma(\xi,\zeta,\mathbf{y})|\zeta-\xi|^{\kappa_{F1}}\eta_\rho''(\zeta-\xi)\,\mathrm{d}\zeta\mathrm{d}\xi.
\end{aligned}
$$

From the bounds on Γ and η''_ρ, by changing variable $\zeta' = \zeta - \xi$, we further have

$$\begin{aligned}
&\int_u^\infty \int_{-\infty}^v \Gamma(\xi,\zeta,\mathbf{y})|\zeta-\xi|^{\kappa_{F1}}\eta''_\rho(\zeta-\xi)\,\mathrm{d}\zeta\mathrm{d}\xi \\
&= \int_u^\infty \int_{-\infty}^{v-\xi} \Gamma(\xi,\zeta'+\xi,\mathbf{y})|\zeta'|^{\kappa_{F1}}\eta''_\rho(\zeta')\,\mathrm{d}\zeta'\mathrm{d}\xi \\
&\le C\rho^{\kappa_{F1}} \int_u^\infty \sup_{\{|\zeta'|<\rho,\,\zeta'<v-\xi\}} \Gamma(\xi,\zeta'+\xi,\mathbf{y})\,\mathrm{d}\xi \\
&\le C\rho^{\kappa_{F1}} \int_u^{v-\rho} \big(|v|^{p-1}+|\xi|^{p-1}+1\big)\,\mathrm{d}\xi \\
&\le C\rho^{\kappa_{F1}}\big(|(u,v)|^p+1\big).
\end{aligned} \tag{3.58}$$

Now applying bound (3.58) yields

$$\begin{aligned}
&\left|\int_0^t \int \bar{H}(\zeta-v)H(\xi-u)\big(\mathbf{F}_u(\zeta,\mathbf{y})-\mathbf{F}_u(\xi,\mathbf{y})\big)\cdot\nabla_{\mathbf{y}}\varphi\,\mathrm{d}E\mathrm{d}s\right| \\
&\le C\int_0^t \iint \big(|(u,v)|^p+1\big)\rho^{\kappa_{F1}}\theta^{-1}J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s \\
&\le C\rho^{\kappa_{F1}}\theta^{-1}\int_0^t \int \big(|(u,v)|^p+1\big)\,\mathrm{d}\mathbf{x}\mathrm{d}s.
\end{aligned} \tag{3.59}$$

For the next integral, we use the fact that $\varphi_\zeta = -\varphi_\xi$:

$$\begin{aligned}
&\left|\int \bar{H}(\zeta-v)H(\xi-u)\big(D_{\mathbf{y}}\cdot\mathbf{F}(\zeta,\mathbf{y})-D_{\mathbf{x}}\cdot\mathbf{F}(\zeta,\mathbf{x})\big)\,\varphi_\zeta\,\mathrm{d}E\right| \\
&= \left|\int \bar{H}(\zeta-v)\,\partial_\xi H(\xi-u)\big(D_{\mathbf{y}}\cdot\mathbf{F}(\zeta,\mathbf{y})-D_{\mathbf{x}}\cdot\mathbf{F}(\zeta,\mathbf{x})\big)\,\varphi\,\mathrm{d}E\right| \\
&\le \theta^{\kappa_{F2}}\iint\int \bar{H}(\zeta-v)(|\zeta|^q+1)\,\eta''_\rho(\zeta-u)\,\mathrm{d}\zeta\;J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\zeta\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&\le C\theta^{\kappa_{F2}}\int \big(|(u,v)|^q+1\big)\,\mathrm{d}\mathbf{x}.
\end{aligned} \tag{3.60}$$

Furthermore, we have

$$\begin{aligned}
&\left|\int_0^t \int \bar{H}(\zeta-v)H(\xi-u)\big(\mathbf{F}_u(\xi,\mathbf{y})-\mathbf{F}_u(\xi,\mathbf{x})\big)\cdot\nabla_{\mathbf{y}}\varphi\,\mathrm{d}E\mathrm{d}s\right| \\
&\le C\|D_{\mathbf{x}}\cdot\mathbf{F}_u\|_{L^\infty}\int_0^t \int \bar{H}(\zeta-v)H(\xi-u)\eta''_\rho(\zeta-\xi)J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}E\mathrm{d}s \\
&\le C\|D_{\mathbf{x}}\cdot\mathbf{F}_u\|_{L^\infty}\int_0^t \iint \eta_\rho(v(\mathbf{y},s)-u(\mathbf{x},s))J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s.
\end{aligned} \tag{3.61}$$

Following (3.61) above, we again have

$$\Big|\int_0^t \int \bar{H}(\zeta - v)H(\xi - u)\big(D_{\mathbf{x}} \cdot \mathbf{F}(\zeta, \mathbf{x}) - D_{\mathbf{x}} \cdot \mathbf{F}(\xi, \mathbf{x})\big)\ \varphi_\zeta \ \mathrm{d}E\mathrm{d}s\Big|$$
$$\le C\int_0^t \int \|D_{\mathbf{x}} \cdot \mathbf{F}_u\|_{L^\infty} \bar{H}(\zeta - v)H(\xi - u)\eta_\rho''(\zeta - \xi)J_\theta(\mathbf{y} - \mathbf{x})\ \mathrm{d}E\mathrm{d}s$$
$$= C\|D_{\mathbf{x}} \cdot \mathbf{F}_u\|_{L^\infty} \int_0^t \iint \eta_\rho(v(\mathbf{y}, s) - u(\mathbf{x}, s))J_\theta(\mathbf{y} - x)\ \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s. \qquad (3.62)$$

Finally, using $\nabla_{\mathbf{x}}\varphi + \nabla_{\mathbf{y}}\varphi = 0$ and adding (3.56)–(3.62) together, we obtain

$$\Big|\int_0^t \int \bar{H}(\zeta - v)H(\xi - u)\big(\mathbf{F}_u(\xi, \mathbf{x}) \cdot \nabla_{\mathbf{x}}\varphi + \mathbf{G}_u(\zeta, \mathbf{y}) \cdot \nabla_{\mathbf{y}}\varphi\big)\ \mathrm{d}E\mathrm{d}s\Big|$$
$$+ \Big|\int_0^t \int \bar{H}(\zeta - v)H(\xi - u)\left(D_{\mathbf{x}} \cdot \mathbf{F}(\xi, \mathbf{x}) - D_{\mathbf{y}} \cdot \mathbf{G}(\zeta, \mathbf{y})\right)\varphi_\xi\ \mathrm{d}E\mathrm{d}s\Big|$$
$$\le C\big(\|\mathbf{G}_u - \mathbf{F}_u\|_{L^\infty}\theta^{-1} + \|D_{\mathbf{x}} \cdot (\mathbf{G} - \mathbf{F})\|_{L^\infty}\rho^{-1}\big)$$
$$\times \int_0^t \iint \eta_\rho(v - u)J_\theta(\mathbf{y} - \mathbf{x})\ \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s$$
$$+ C\|D_{\mathbf{x}} \cdot \mathbf{F}_u\|_{L^\infty} \int_0^t \iint \eta_\rho(v(\mathbf{y}, s) - u(\mathbf{x}, s))J_\theta(\mathbf{y} - \mathbf{x})\ \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s$$
$$+ C\big(\rho^{\kappa_{F1}}\theta^{-1} + \theta^{\kappa_{F2}}\big)\int_0^t \int \big(|(u, v)|^p + |(u, v)|^q + 1\big)\ \mathrm{d}\mathbf{x}\mathrm{d}s.$$

3. *Itô correction term.* The Itô correction integral can be estimated as follows:

$$\mathbb{E}[I^\sigma] = \frac{1}{2}\mathbb{E}\big[\int_0^t \iint \big(\tau(v(\mathbf{y}, s)) - \sigma(u(\mathbf{x}, s))\big)^2\varphi(u(\mathbf{x}, s), v(\mathbf{y}, s), \mathbf{x}, \mathbf{y})\ \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s\big]$$
$$\le \mathbb{E}\big[\int_0^t \iint \big((\tau(v) - \sigma(v))^2 + (\sigma(v) - \sigma(u))^2\big)\ \eta_\rho''(v - u)J_\theta(\mathbf{y} - \mathbf{x})\ \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s\big]$$
$$\le \mathbb{E}\big[\int_0^t \iint \big(\|\tau - \sigma\|^2_{L^\infty} + \frac{(\sigma(v) - \sigma(u))^2}{|v - u|^{2\lambda_\sigma}}\rho^{2\lambda_\sigma}\big)\eta_\rho''(v - u)J_\theta(\mathbf{y} - \mathbf{x})\ \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s\big]$$
$$\le C\rho^{-1}\|\tau - \sigma\|^2_{L^\infty} \int_0^t \iint J_\theta(\mathbf{y} - \mathbf{x})\ \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}\mathrm{d}s$$
$$+ C_\sigma\rho^{2\lambda_\sigma}\mathbb{E}\big[\int_0^t \iint \eta_\rho''(v - u)J_\theta(\mathbf{y} - \mathbf{x})\ \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s\big]$$
$$\le Ct\left(\rho^{-1}\|\tau - \sigma\|^2_{L^\infty} + \rho^{2\lambda_\sigma - 1}\right)|\mathbb{T}^d|.$$

In the above, $\sigma(u)$ and $\tau(v)$ are symmetric.

4. *Mollification term.* We now follow the argument in the proof of [15, Theorem 3.2] closely to obtain the estimate in this step, which will be further refined in Corollary 4.2.

First we decompose the difference:

$$\begin{aligned} I^\eta &= \iint f^+(\xi,\mathbf{x},t)\bar{g}^+(\xi,\mathbf{x},t)\,\mathrm{d}\xi\,\mathrm{d}\mathbf{x} - \int f^+(\xi,\mathbf{x},t)\bar{g}^+(\zeta,\mathbf{y},t)\varphi(\xi,\zeta,\mathbf{x},\mathbf{y})\,\mathrm{d}E \\ &= I_1^\eta(t) + I_2^\eta(t), \end{aligned}$$

where

$$\begin{aligned} I_1^\eta(t) := &\iint f^+(\xi,\mathbf{x},t)\bar{g}^+(\xi,\mathbf{x},t)\,\mathrm{d}\xi\,\mathrm{d}\mathbf{x} \\ &- \iint\int f^+(\xi,\mathbf{x},t)\bar{g}^+(\xi,\mathbf{y},t)J_\theta(\mathbf{x}-\mathbf{y})\,\mathrm{d}\xi\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}, \end{aligned}$$

and

$$\begin{aligned} I_2^\eta(t) := &\iint\int f^+(\xi,\mathbf{x},t)\bar{g}^+(\xi,\mathbf{y},t)J_\theta(\mathbf{x}-\mathbf{y})\,\mathrm{d}\xi\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\ &- \int f^+(\xi,\mathbf{x},t)\bar{g}^+(\zeta,\mathbf{y},t)\varphi(\xi,\zeta,\mathbf{x},\mathbf{y})\,\mathrm{d}E. \end{aligned}$$

We first have

$$\begin{aligned} |I_1^\eta| &= \Big|\iint f^+(\xi,\mathbf{x},t)\Big(\bar{g}^+(\xi,\mathbf{x},t) - \int \bar{g}^+(\xi,\mathbf{y},t)J_\theta(\mathbf{x}-\mathbf{y})\,\mathrm{d}y\Big)\,\mathrm{d}\xi\mathrm{d}\mathbf{x}\Big| \\ &\le \iint \Big|\chi_{\bar{g}^+}(\xi,\mathbf{x},t) - \int \chi_{\bar{g}^+}(\xi,\mathbf{y},t)J_\theta(\mathbf{x}-\mathbf{y})\,\mathrm{d}y\Big|\,\mathrm{d}\xi\mathrm{d}\mathbf{x}, \end{aligned}$$

where $\chi_{\bar{g}^+} = \bar{g}^+(\xi,\mathbf{x},t) - \mathbb{1}_{\xi<0}$ is integrable in both ξ and $\mathbf{x}$, which is the kinetic functions used in, e.g., [9]. From the Lebesgue differentiation theorem, the absolute value in the last line tends to zero as $\theta \to 0$, except on an $\mathbf{x}$-measure zero set. Since $\chi_{\bar{g}^+}$ is integrable in both ξ and $\mathbf{x}$, we can take this limit outside the $(\xi,\mathbf{x})$-integrals. This implies that

$$\lim_{\theta\to 0}|I_1^\eta| = 0 \qquad \text{almost surely.} \tag{3.63}$$

For any test function $\psi \in C_0^\infty(\mathbb{R})$, we have

$$\begin{aligned} &\int\iint \psi''(\zeta-\xi)f^+(\xi,\mathbf{x},t)\bar{g}^+(\zeta,\mathbf{x},t)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}\mathbf{x} \\ &= -\int\iint \psi(\zeta-\xi)\mathrm{d}_\xi f^+(\xi,\mathbf{x},t)\mathrm{d}_\zeta\bar{g}^+(\zeta,\mathbf{x},t)\,\mathrm{d}\mathbf{x}, \end{aligned}$$

where both $\mathrm{d}_\xi f^+(\xi,\mathbf{x},t)$ and $-\mathrm{d}_\zeta\bar{g}^+(\zeta,\mathbf{x},t)$ have the unit mass for each $(\mathbf{x},t)$.

From our construction of η_ρ in (3.8)–(3.9), we see that, for $\varpi > -\rho$,

$$(\varpi+\rho)_+ - \eta_\rho(\varpi) = \int_{-\rho}^{\varpi}\int_r^\infty \eta_\rho''(s)\,\mathrm{d}s\,\mathrm{d}r = \int_{-1}^\infty \big((\varpi+\rho)_+ \wedge (s\rho+\rho)\big)\eta''(s)\,\mathrm{d}s.$$

Taking $\varpi = \zeta - \xi$, we find that, for any $s > -1$,

$$\begin{aligned}(\zeta + \rho - \xi)_+ \wedge (s\rho + \rho) &\le (\zeta + \rho)_+ \wedge (s\rho + \rho) + (\xi)_+ \wedge (s\rho + \rho) \\ &\le \big((\zeta + \rho)_+ + (\xi)_+\big) \wedge \big(2\rho(s+1)\big).\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}0 &\le \iint f^+(\xi, \mathbf{x}, t)\bar{g}^+(\xi - \rho, \mathbf{y}, t)\, \mathrm{d}\xi - \iint \eta_\rho''(\zeta - \xi) f^+(\xi, \mathbf{x}, t)\bar{g}^+(\zeta, \mathbf{y}, t)\, \mathrm{d}\xi\, \mathrm{d}\zeta \\ &= -\iint \big((\zeta + \rho - \xi)_+ - \eta_\rho(\zeta - \xi)\big)\mathrm{d}_\xi f^+(\xi, \mathbf{x}, t)\mathrm{d}_\zeta \bar{g}^+(\zeta, \mathbf{y}, t) \\ &\le \int_{-1}^{\infty} \big(\big|\int (\xi)_+ \mathrm{d}_\xi f^+(\xi, \mathbf{x}, t)\big| + \big|\int (\zeta + \rho)_+ \mathrm{d}_\zeta \bar{g}^+(\zeta, \mathbf{y}, t)\big|\big) \wedge \big(2\rho(s+1)\big)\, \eta''(s)\, \mathrm{d}s \\ &\le 2\rho \int_{-1}^{\infty} (s+1)\eta''(s)\, \mathrm{d}s \\ &\le C\rho,\end{aligned}$$

because $\eta''(s)$ is supported on $[-1, 1]$. Integrating the foregoing against $J_\theta(\mathbf{x} - y)\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y}$ yields that

$$|I_2^\eta| \le C\rho \iint J_\theta(\mathbf{x} - \mathbf{y})\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y} = C\rho, \tag{3.64}$$

where constant C is in fact deterministic.

Putting together (3.63) and (3.64), we obtain that

$$\lim_{\theta, \rho \to 0} \left| \iint f^+(\xi, \mathbf{x}, t)\bar{g}^+(\xi, \mathbf{x}, t)\, \mathrm{d}\xi\, \mathrm{d}\mathbf{x} - \int f^+(\xi, \mathbf{x}, t)\bar{g}^+(\zeta, \mathbf{y}, t)\varphi(\xi, \zeta, \mathbf{x}, \mathbf{y})\, \mathrm{d}E \right| = 0$$

almost surely. From the uniform boundedness of $\|J_\theta\|_{L^1(\mathbb{T}^d)}$ and $\|\eta_\rho\|_{L^1(\mathbb{R})}$, we can move the limit outside the expectation and conclude that there exists $r = r_t(\theta, \rho)$ such that, for any fixed t,

$$\mathbb{E}[|I^\eta|] \le r_t(\theta, \rho) \to 0 \qquad \text{as } \theta, \rho \to 0.$$

This leads to estimate (3.46) and completes the proof.

Remark 3.1. In the whole-space case (*i.e.*, $\mathbb{T}^d$ is replaced by $\mathbb{R}^d$), it is necessary to modify the test function to $\varphi(\xi, \zeta, \mathbf{x}, \mathbf{y}) = \eta_\rho''(\zeta - \xi)J_\theta(\mathbf{y} - \mathbf{x})\psi(\frac{\mathbf{y}+\mathbf{x}}{2})$, where ψ is a non-negative smooth function $\mathbb{R}^d \to \mathbb{R}$ supported on $B_R(\mathbf{0})$. The terms involving $\|\nabla\psi\|_{L^\infty}$ and $\|\psi\|_{L^1}$ appear respectively in the parabolic and Itô correction terms. In particular, in the Itô correction and mollification estimates, $\rho\|\psi\|_{L^1}$ is involved so that, for $R \to \infty$ (so that $\psi \to f(\mathbf{x}) \equiv 1$ pointwise), one needs $\rho \to 0$ first; otherwise, no estimates are possible without prescribing very specific forms for ψ, or we need to consider the weighted space $L^1(\psi(\mathbf{x})\,\mathrm{d}\mathbf{x})$.

4. L^1–Stability Estimate

In this section, we establish the following L^1–stability theorem.

Theorem 4.1 (L^1–stability estimate)**.** *Let u and v be stochastic kinetic solutions of* (1.1) *with initial data u_0 and v_0, respectively. Let the nonlinear functions of* (1.1) *satisfy assumptions* (1.6)–(1.10) *with $\lambda_\sigma > \frac{1}{2}$. Then the following L^1–stability estimate holds*:

$$\mathbb{E}\Big[\int (v^+(\mathbf{x},t)-u^+(\mathbf{x},t))_+ \,\mathrm{d}\mathbf{x}\Big] \le \exp\big\{C\|D_\mathbf{x}\cdot\mathbf{F}_u\|_{L^\infty} t\big\}\,\mathbb{E}\Big[\int (v_0(\mathbf{x})-u_0(\mathbf{x}))_+ \,\mathrm{d}\mathbf{x}\Big], \tag{4.1}$$

where C is a constant depending only on d.

Proof. In this case, $\mathbf{F}(\cdot,\cdot)=\mathbf{G}(\cdot,\cdot)$, $\mathbf{A}(\cdot)=\mathbf{B}(\cdot)$, and $\sigma(\cdot)=\tau(\cdot)$. Then, from Proposition 3.2, we obtain

$$\begin{aligned}
&\mathbb{E}[I^a] \le C(\boldsymbol{\alpha})\rho^{2\gamma_\alpha}\theta^{-2}\mathbb{E}\Big[\int_0^t \iint \eta_\rho(v-u)\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s\Big],\\
&\mathbb{E}[I^F] \le C\|D_\mathbf{x}\cdot\mathbf{F}_u\|_{L^\infty}\mathbb{E}\Big[\int_0^t \iint \eta_\rho(v(\mathbf{y})-u(\mathbf{x}))J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s\Big]\\
&\qquad\qquad + C\big(\rho^{\kappa_{F1}}\theta^{-1}+\theta^{\kappa_{F2}}\big)\mathbb{E}\Big[\int_0^t\int \big(|(u,v)|^p+|(u,v)|^q+1\big)\,\mathrm{d}\mathbf{x}\mathrm{d}s\Big],\\
&\mathbb{E}[I^\sigma] \le C_\sigma t\rho^{2\lambda_\sigma-1},\\
&\mathbb{E}[I^\eta] = o_{\theta,\rho}(1)\to 0 \qquad \text{as } \theta,\rho\to 0.
\end{aligned}$$

Taking the limits in the order: $\rho\to 0$ first and $\theta\to 0$ second, we obtain estimate:

$$\begin{aligned}
&\mathbb{E}\Big[\iint f^+(\xi,\mathbf{x},t)\bar{g}^+(\xi,\mathbf{x},t)\,\mathrm{d}\xi\,\mathrm{d}\mathbf{x}\Big]\\
&\le \mathbb{E}\Big[\int f^+(\xi,\mathbf{x},0)\bar{g}^+(\xi,\mathbf{x},0)\,\mathrm{d}\mathbf{x}\Big]\\
&\quad + C\|D_\mathbf{x}\cdot\mathbf{F}_u\|_{L^\infty}\mathbb{E}\Big[\int_0^t\iint f^+(\xi,\mathbf{x},s)\bar{g}^+(\xi,\mathbf{x},s)\,\mathrm{d}\xi\mathrm{d}\mathbf{x}\mathrm{d}s\Big]
\end{aligned}\tag{4.2}$$

for every $t\in[0,T]$.

Taking $f(\xi,\mathbf{x},0)=H(\xi-u_0)$ and $\bar{g}(\zeta,\mathbf{y},0)=\bar{H}(\zeta-v_0)$, we can argue exactly as in [15, Proposition 2.11] and conclude that the kinetic measure does not concentrate at $t=0$, $\mathbb{P}$-almost surely, and that $f^+(\xi,\mathbf{x},0)=f(\xi,\mathbf{x},0)$ and $\bar{g}^+(\zeta,\mathbf{y},0)=\bar{g}(\zeta,\mathbf{y},0)$.

Taking $u_0=v_0$ that leads to

$$\mathbb{E}\Big[\int f^+(\xi,\mathbf{x},0)\bar{g}^+(\xi,\mathbf{x},0)\,\mathrm{d}\mathbf{x}\Big]=0,$$

we see from (4.2) that

$$
\begin{aligned}
&\mathbb{E}\big[\iint f^+(\xi,\mathbf{x},t)\bar{g}^+(\xi,\mathbf{x},t)\,\mathrm{d}\xi\,\mathrm{d}\mathbf{x}\big]\\
&\le C\|D_{\mathbf{x}}\cdot\mathbf{F}_u\|_{L^\infty}\int_0^t \mathbb{E}\big[\iint f^+(\xi,\mathbf{x},s)\bar{g}^+(\xi,\mathbf{x},s)\,\mathrm{d}\xi\mathrm{d}\mathbf{x}\big]\mathrm{d}s
\end{aligned}
$$

for every $t\in[0,T]$. Then it follows from the Gronwall inequality that

$$
\mathbb{E}\big[\iint f^+(\xi,\mathbf{x},t)\bar{g}^+(\xi,\mathbf{x},t)\,\mathrm{d}\xi\,\mathrm{d}\mathbf{x}\big]=0,
$$

which implies that, for every $t\in[0,T]$,

$$
f^+(\xi,\mathbf{x},t)\big(1-f^+(\xi,\mathbf{x},t)\big)=0 \qquad (\omega,\xi,\mathbf{x})\text{-almost everywhere.}
$$

This means that, for every $t\in[0,T]$, f^+ takes values in $\{0,1\}$ $(\omega,\xi,\mathbf{x})$-almost everywhere. Similarly, it can be shown that, for every $t\in[0,T]$, $\bar{g}^+$ also takes values in $\{0,1\}$ $(\omega,\xi,\mathbf{x})$-almost everywhere. This allows us to represent f and $\bar{g}$ as the Heaviside functions inside the integrals and to write

$$
\mathbb{E}\big[\iint f^+(\xi,\mathbf{x},t)\bar{g}^+(\xi,\mathbf{x},t)\,\mathrm{d}\xi\,\mathrm{d}\mathbf{x}\big]=\mathbb{E}\big[\iint (v^+(\mathbf{x},t)-u^+(\mathbf{x},t))_+\,\mathrm{d}\xi\,\mathrm{d}\mathbf{x}\big].
$$

Then bound (4.1) follows directly from 4.2 via the Gronwall inequality.

It follows from Lemma 3.1 as in [15, Corollary 3.3] (also see [14, Corollary 12] and [13, Corollary 3.4]) that

Corollary 4.1. *Let u be a stochastic kinetic solution to* (1.1)*. There exists a version of u with almost surely continuous paths in $L^p(\mathbb{T}^d)$.*

In particular, from the proof of Proposition 4.1 and Lemma 3.1, at each $t\in[0,T]$, f^+ can be represented $(\omega,\xi,\mathbf{x})$–almost everywhere as $H(\xi-u^+)$ and is right-continuous and, similarly, f^- can be represented as $H(\xi-u^-)$, $(\omega,\xi,\mathbf{x})$–almost everywhere and is left-continuous. From (3.12), it must be that $u^+=u^-$ $\mathbb{P}$-almost surely, and u is continuous $[0,T]\to L^p(\mathbb{T}^d)$.

From now on, we can drop the distinction between $u^\pm$ and u (resp. $v^\pm$ and v) and simply refer to u (resp. v); we can also write $f^+(\xi,\mathbf{x},t)$ as $H(\xi-u(\mathbf{x},t))$ (resp. $\bar{g}^+(\zeta,\mathbf{y},t)$ as $\bar{H}(\zeta-v(\mathbf{y},t))$).

Remark 4.1. If $\mathbf{F}$ is space-translational invariant (so that $D_{\mathbf{x}}\cdot\mathbf{F}_u=0$), then we conclude the familiar L^1–contraction estimate:

$$
\mathbb{E}\big[\int (v(\mathbf{x},t)-u(\mathbf{x},t))_+\,\mathrm{d}\mathbf{x}\big]\le\mathbb{E}\big[\int (v_0(\mathbf{x})-u_0(\mathbf{x}))_+\,\mathrm{d}\mathbf{x}\big].
$$

Corollary 4.2. *With I^η defined as in* (3.29), *we in fact have the bound*:

$$\mathbb{E}[I^\eta] \le \sup_{|\mathbf{h}|<\theta} \mathbb{E}\big[\int (u(\mathbf{x},t) - u(\mathbf{x}+\mathbf{h},t))_+ \,\mathrm{d}\mathbf{x}\big]. \tag{4.3}$$

Proof. We can now write I^η as

$$\begin{aligned}
I^\eta &= \int (v(\mathbf{x},t) - u(\mathbf{x},t))_+ \,\mathrm{d}\mathbf{x} - \int H(\xi - u(\mathbf{x},t))\bar{H}(\zeta - v(\mathbf{y},t))\varphi(\xi,\zeta,\mathbf{x},\mathbf{y}) \,\mathrm{d}E \\
&= \int (v(\mathbf{x},t) - u(\mathbf{x},t))_+ \,\mathrm{d}\mathbf{x} - \iint \eta_\rho(v(\mathbf{y},t) - u(\mathbf{x},t)) J_\theta(\mathbf{x}-\mathbf{y}) \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}.
\end{aligned}$$

Using the basic inequality $(\cdot)_+ \le \eta_\rho(\cdot)$, we obtain

$$\begin{aligned}
&\mathbb{E}\big[\int (v(\mathbf{x},t) - u(\mathbf{x},t))_+ \,\mathrm{d}\mathbf{x}\big] \\
&= \mathbb{E}\big[\iint (v(\mathbf{y},t) - u(\mathbf{y},t))_+ J_\theta(\mathbf{y}-\mathbf{x}) \,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\big] \\
&\le \mathbb{E}\big[\iint (v(\mathbf{y},t) - u(\mathbf{x},t))_+ J_\theta(\mathbf{y}-\mathbf{x}) \,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\big] \\
&\quad + \mathbb{E}\big[\iint (u(\mathbf{x},t) - u(\mathbf{y},t))_+ J_\theta(\mathbf{y}-\mathbf{x}) \,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\big] \\
&\le \mathbb{E}\big[\iint \eta_\rho(v(\mathbf{y},t) - u(\mathbf{x},t)) J_\theta(\mathbf{y}-\mathbf{x}) \,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\big] \\
&\quad + \mathbb{E}\big[\iint (u(\mathbf{x},t) - u(\mathbf{y},t))_+ J_\theta(\mathbf{y}-\mathbf{x}) \,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\big].
\end{aligned}$$

Set $\mathbf{h} := \mathbf{y} - \mathbf{x}$. Then we have

$$\begin{aligned}
&\mathbb{E}\big[\iint (u(\mathbf{x},t) - u(\mathbf{y},t))_+ J_\theta(\mathbf{y}-\mathbf{x}) \,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\big] \\
&= \mathbb{E}\big[\int \big(\int (u(\mathbf{x},t) - u(\mathbf{x}+\mathbf{h},t))_+ \,\mathrm{d}\mathbf{x}\big) J_\theta(\mathbf{h}) \,\mathrm{d}\mathbf{h}\big] \\
&\le \sup_{|\mathbf{h}|<\theta} \mathbb{E}\big[\int (u(\mathbf{x},t) - u(\mathbf{x}+\mathbf{h},t))_+ \,\mathrm{d}\mathbf{x}\big] \int J_\theta(\mathbf{h}) \,\mathrm{d}\mathbf{h} \\
&= \sup_{|\mathbf{h}|<\theta} \mathbb{E}\big[\int (u(\mathbf{x},t) - u(\mathbf{x}+\mathbf{h},t))_+ \,\mathrm{d}\mathbf{x}\big],
\end{aligned}$$

where we have used that $\int J_\theta(\mathbf{h}) \,\mathrm{d}\mathbf{h} = 1$. This completes the proof.

5. Fractional BV Estimate

We now apply Proposition 3.2 to the pair of two equations:

$$\partial_t u = -\nabla\cdot\mathbf{F}(u,\mathbf{x}) + \nabla\cdot(\mathbf{A}(u)\nabla u) + \sigma(u)\dot{W}, \tag{5.1}$$

$$\partial_t v = -\nabla\cdot\mathbf{F}(v,\mathbf{x}+\mathbf{h}) + \nabla\cdot(\mathbf{A}(v)\nabla v) + \sigma(v)\dot{W}, \tag{5.2}$$

with initial conditions $u(\mathbf{x},0)=u_0(\mathbf{x})$ and $v(\mathbf{x},0)=u_0(\mathbf{x}+\mathbf{h})$, respectively, and derive a fractional BV estimate. In this case, $\gamma_{\boldsymbol{\alpha}}=\gamma_{\boldsymbol{\beta}}=\gamma$ and $\lambda_\sigma=\lambda_\tau=\lambda$. With this fractional BV estimate, we can also refine our continuous dependence estimate.

Theorem 5.1 (Fractional BV estimate)**.** *Let u be a stochastic kinetic solution of* (1.1) *with initial data u_0. Let the nonlinear functions of* (1.1) *satisfy assumptions* (1.6)–(1.10) *with $\lambda_\sigma>\frac{1}{2}$. Then the following fractional BV estimate holds*:

$$\begin{aligned}
&\mathbb{E}\big[\int (u(\mathbf{x}+\mathbf{h},t)-u(\mathbf{x},t))_+\,\mathrm{d}\mathbf{x}\big]\\
&\le \exp\big\{C\|D_{\mathbf{x}}\cdot\mathbf{F}\|_{L^\infty}t\big\}\\
&\quad\times\Big(\hat{K}_1(\mathbf{F},t)|\mathbf{h}|^{\kappa_{F2}}+\mathbb{E}\big[\iint (u_0(\mathbf{y}+\mathbf{h})-u_0(\mathbf{x}))_+J_{|\mathbf{h}|}(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\big]\Big),
\end{aligned}\tag{5.3}$$

where C depends on d, and $\hat{K}_1(\mathbf{F},t)$ depends on $\|(u_0,v_0)\|_{L^p}$ and is proportional to the Hölder norm of $D_{\mathbf{x}}\cdot\mathbf{F}(\cdot,\mathbf{x})$ in $\mathbf{x}$.

In particular, if u_0 is in the κ_{F2}–Nikolskii space with $\kappa_{F2}\le 1$, i.e., the functions of bounded κ_{F2}^{-1} variation, then the fractional BV bound holds:

$$\mathbb{E}\big[|u|_{N^{\kappa_{F2},1}}(t)\big]\le\exp\big\{C\|D_{\mathbf{x}}\cdot\mathbf{F}_u\|_{L^\infty}t\big\}\big(\hat{K}_1(\mathbf{F},t)+E\big[|u_0|_{N^{\kappa_{F2},1}}\big]\big),$$

where $|\cdot|_{N^{\kappa,1}}$ denotes the bounded $\frac{1}{\kappa}$–variation semi-norm, the Nikolskii semi-norm (1.5)*.*

Proof. We first notice that, if $u(\mathbf{x},t)$ solves (5.1) with the initial data $u_0(\mathbf{x})$, then $v(\mathbf{z},t)=u(\mathbf{x}+\mathbf{h},t)$ for $\mathbf{z}=\mathbf{x}+\mathbf{h}$ solves (5.2) with the initial data $u_0(\mathbf{x}+\mathbf{h})$.

As in the L^1–stability estimate in §4, choosing $\mathbf{B}=\mathbf{A}$ and $\tau=\sigma$ in Proposition 3.2 and using Corollary 4.2, we have

$$\mathbb{E}[I^a]\le C(\boldsymbol{\alpha})\rho^{2\gamma}\theta^{-2}\mathbb{E}\big[\int_0^t\iint\eta_\rho(v-u)J_\theta(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s\big], \tag{5.4}$$

$$\mathbb{E}[I^\sigma]\le C_\sigma\rho^{2\lambda-1}, \tag{5.5}$$

$$\mathbb{E}[I^\eta]\le\sup_{|\mathbf{h}|\le\theta}\mathbb{E}\big[\int(u(\mathbf{x}+\mathbf{h},t)-u(\mathbf{x},t))_+\,\mathrm{d}\mathbf{x}\big]. \tag{5.6}$$

Choosing $\mathbf{G}(\cdot,\cdot)=\mathbf{F}(\cdot,\cdot+\mathbf{h})$, we see from (3.54) and assumptions (1.6)–(1.8) that

$$
\begin{aligned}
&\big|\mathbf{F}_u(\xi,\mathbf{x})\cdot\nabla_{\mathbf{x}}\varphi+\mathbf{G}_u(\zeta,\mathbf{y})\cdot\nabla_{\mathbf{y}}\varphi\big|\\
&\le\big|\big(\mathbf{F}_u(\zeta,\mathbf{y}+\mathbf{h})-\mathbf{F}_u(\zeta,\mathbf{y})\big)\cdot\nabla_{\mathbf{y}}\varphi\big|+\big|\big(\mathbf{F}_u(\zeta,\mathbf{y})-\mathbf{F}_u(\xi,\mathbf{y})\big)\cdot\nabla_{\mathbf{y}}\varphi\big|\\
&\quad+\big|\big(\mathbf{F}_u(\xi,\mathbf{y})-\mathbf{F}_u(\xi,\mathbf{x})\big)\cdot\nabla_{\mathbf{y}}\varphi\big|+\big|\mathbf{F}_u(\xi,\mathbf{x})\cdot(\nabla_{\mathbf{x}}\varphi+\nabla_{\mathbf{y}}\varphi)\big|\\
&\le C\left\|D_{\mathbf{x}}\cdot\mathbf{F}_u\right\|_{L^\infty}|\mathbf{h}|\vartheta^{-1}\varphi+C|(\xi,\zeta)|^{p-1}\rho^{\kappa_{F1}}\theta^{-1}\varphi+C\left\|D_{\mathbf{x}}\cdot\mathbf{F}_u\right\|_{L^\infty}\varphi,
\end{aligned}
$$

where we have used $|\nabla_{\mathbf{y}}\varphi|\le C\theta^{-1}\varphi$ and $\nabla_{\mathbf{x}}\varphi+\nabla_{\mathbf{y}}\varphi=0$.

Furthermore, using (3.60),

$$
\begin{aligned}
&\left|\int H(\xi-u)\bar{H}(\zeta-v)\big(D_{\mathbf{y}}\cdot\mathbf{G}(\zeta,\mathbf{y})-D_{\mathbf{x}}\cdot\mathbf{F}(\xi,\mathbf{x})\big)\varphi_\zeta\,\mathrm{d}E\right|\\
&\le\left|\int H(\xi-u)\bar{H}(\zeta-v)\big(D_{\mathbf{y}}\cdot\mathbf{F}(\zeta,\mathbf{y}+\mathbf{h})-D_{\mathbf{x}}\cdot\mathbf{F}(\xi,\mathbf{y}+\mathbf{h})\big)\varphi_\zeta\,\mathrm{d}E\right|\\
&\quad+\left|\int H(\xi-u)\bar{H}(\zeta-v)\big(D_{\mathbf{y}}\cdot\mathbf{F}(\xi,\mathbf{y}+\mathbf{h})-D_{\mathbf{x}}\cdot\mathbf{F}(\xi,\mathbf{x})\big)\varphi_\zeta\,\mathrm{d}E\right|\\
&\le C\left\|D_{\mathbf{x}}\cdot\mathbf{F}\right\|_{L^\infty}\left|\int H(\xi-u)\bar{H}(\zeta-v)\varphi(\xi,\zeta,\mathbf{x},\mathbf{y})\,\mathrm{d}E\right|\\
&\quad+C(\theta+|\mathbf{h}|)^{\kappa_{F2}}\int\big(|(u,v)|^q+1\big)\,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

Let $|\mathbf{h}|,\theta<1$. Then

$$
\begin{aligned}
|I^F|\le{}&C\|D_{\mathbf{x}}\cdot\mathbf{F}_u\|_{L^\infty}\int_0^t\iint\eta_\rho(v(\mathbf{y},s)-u(\mathbf{x},s))\big(|\mathbf{h}|\theta^{-1}+1\big)J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s\\
&+C\big(\rho^{\kappa_{F1}}\theta^{-1}+(\theta+|\mathbf{h}|)^{\kappa_{F2}}\big)\int_0^t\int\big(|(u,v)|^p+|(u,v)|^q+1\big)\,\mathrm{d}\mathbf{x}\mathrm{d}s.
\end{aligned}
$$

Combining this estimate with (5.4)–(5.6), we have

$$
\begin{aligned}
&\mathbb{E}\big[\iint\eta_\rho(u(\mathbf{y}+\mathbf{h},t)-u(\mathbf{x},t))J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\big]\\
&\le\mathbb{E}\big[\iint\eta_\rho(u_0(\mathbf{y}+\mathbf{h})-u_0(\mathbf{x}))J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\big]\\
&\quad+C\|D_{\mathbf{x}}\cdot\mathbf{F}_u\|_{L^\infty}\big(|\mathbf{h}|\theta^{-1}+1\big)\mathbb{E}\big[\int_0^t\iint\eta_\rho(v-u)J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s\big]\\
&\quad+C\rho^{2\gamma}\theta^{-2}\mathbb{E}\big[\int_0^t\iint\eta_\rho(v-u)J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s\big]\\
&\quad+C\big(\rho^{\kappa_{F1}}\theta^{-1}+(\theta+|\mathbf{h}|)^{\kappa_{F2}}\big)\mathbb{E}\big[\int_0^t\int\big(|(u,v)|^p+|(u,v)|^q+1\big)\,\mathrm{d}\mathbf{x}\mathrm{d}s\big]\\
&\quad+C_\sigma\rho^{2\lambda-1}|\mathbb{T}^d|.
\end{aligned}
$$

Next, we apply the Gronwall inequality and use the estimates on $\mathbb{E}\big[I^\eta\big]$ to conclude the proof by choosing $\theta = |\mathbf{h}|$ and taking $\rho \to 0$.

In particular, if $u(\mathbf{x}, t)$ is in the fractional BV class in $\mathbf{x}$ with index κ for any fixed $t > 0$, then

$$\mathbb{E}\big[\int \big(u(\mathbf{x}+\mathbf{h}, t) - u(\mathbf{x}, t)\big)_+ \,\mathrm{d}\mathbf{x}\big] \leq C|\mathbf{h}|^\kappa,$$

which is equivalent to

$$\mathbb{E}\big[\int \big(u(\mathbf{y}+\mathbf{h}, t) - u(\mathbf{x}, t)\big)_+ J_{|\mathbf{h}|}(\mathbf{x}-\mathbf{y}) \,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\big] \leq C|\mathbf{h}|^\kappa.$$

This can be seen as follows: If $\mathbb{E}\big[\int \big(u(\mathbf{x}+\mathbf{h}, t) - u(\mathbf{x}, t)\big)_+ \,\mathrm{d}\mathbf{x}\big] \leq C|\mathbf{h}|^\kappa$, then

$$\begin{aligned}
&\mathbb{E}\big[\iint (u(\mathbf{y}+\mathbf{h}, t) - u(\mathbf{x}, t))_+ J_{|\mathbf{h}|}(\mathbf{y}-\mathbf{x}) \,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\big] \\
&\leq \mathbb{E}\big[\iint \big((u(\mathbf{y}+\mathbf{h}, t) - u(\mathbf{x}+\mathbf{h}, t))_+ \\
&\qquad\qquad + (u(\mathbf{x}+\mathbf{h}, t) - u(\mathbf{x}, t))_+\big) J_{|\mathbf{h}|}(\mathbf{y}-\mathbf{x}) \,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\big] \\
&\leq \mathbb{E}\big[\iint \big((u(\mathbf{y}, t) - u(\mathbf{x}, t))_+ J_{|\mathbf{h}|}(\mathbf{y}-\mathbf{x}) \\
&\qquad\qquad + (u(\mathbf{x}+\mathbf{h}, t) - u(\mathbf{x}, t))_+ J_{|\mathbf{h}|}(\mathbf{y}-\mathbf{x})\big) \,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\big] \\
&\leq \sup_{\mathbf{z} \in B_{|\mathbf{h}|}(\mathbf{0})} \mathbb{E}\big[\int (u(\mathbf{x}+\mathbf{z}, t) - u(\mathbf{x}, t))_+ \,\mathrm{d}\mathbf{x} \int J_{|\mathbf{h}|}(\mathbf{y}-\mathbf{x}) \,\mathrm{d}\mathbf{y}\big] + C|\mathbf{h}|^\kappa \leq C|\mathbf{h}|^\kappa.
\end{aligned}$$

Conversely, if $\mathbb{E}\big[\iint (u(\mathbf{y}+\mathbf{h}, t) - u(\mathbf{x}, t))_+ J_{|\mathbf{h}|}(\mathbf{y}-\mathbf{x}) \,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\big] \leq C|\mathbf{h}|^\kappa$, then

$$\begin{aligned}
&\mathbb{E}\big[\int (u(\mathbf{x}+\mathbf{h}, t) - u(\mathbf{x}, t))_+ \,\mathrm{d}\mathbf{x}\big] \\
&\leq \mathbb{E}\big[\iint \big((u(\mathbf{x}+\mathbf{h}, t) - u(\mathbf{y}, t))_+ J_\theta(\mathbf{y}-\mathbf{x}) \\
&\qquad\qquad + (u(\mathbf{y}, t) - u(\mathbf{x}, t))_+ J_\theta(\mathbf{y}-\mathbf{x})\big) \,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\big] \\
&\leq C|\mathbf{h}|^\kappa + \sup_{\mathbf{z} \in B_{|\mathbf{h}|}(\mathbf{0})} \mathbb{E}\big[\iint (u(\mathbf{y}+\mathbf{z}, t) - u(\mathbf{x}, t))_+ J_\theta(\mathbf{y}-\mathbf{x}) \,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\big] \leq C|\mathbf{h}|^\kappa.
\end{aligned}$$

Therefore, if $\mathbb{E}\big[|u_0|_{N^{\kappa,1}}\big] < \infty$, then $\mathbb{E}\big[|u|_{N^{\kappa,1}}\big] < \infty$. This completes the proof.

Remark 5.1. If $\kappa_{F2} = 1$, we obtain an actual BV estimate by taking the supremum (*cf.* [12, Theorem 1.7.2] and [27, Definition 1] for the deterministic case), whilst sending $\theta = |\mathbf{h}| \to 0$. In fact, adding to the inequality by the corresponding inequality for $\big(u(\mathbf{y}+\mathbf{h})-u(\mathbf{x})\big)_-$, we have

$$\mathbb{E}\big[|u|_{BV}(t)\big] \le \exp\big\{C(d)\|D_{\mathbf{x}}\cdot\mathbf{F}_u\|_{L^\infty}t\big\}\big(\hat{K}_1(\mathbf{F},t)+\mathbb{E}\big[|u_0|_{BV}\big]\big). \tag{5.7}$$

Finally, in the space-translational invariant case, $\hat{K}_1(\mathbf{F})=0$ and $\|D_{\mathbf{x}}\cdot\mathbf{F}_u\|_{L^\infty}=0$, so that the classical BV bound follows:

$$\mathbb{E}\big[|u|_{N^{\kappa_{F2}},1}(t)\big] \le \mathbb{E}\big[|u_0|_{N^{\kappa_{F2}},1}\big]. \tag{5.8}$$

In particular, when $\kappa_{F2}=1$,

$$\mathbb{E}\big[|u|_{BV}(t)\big] \le \mathbb{E}\big[|u_0|_{BV}\big]. \tag{5.9}$$

6. Continuous Dependence Estimate

A continuous dependence estimate for equations (3.1)–(3.2) is an estimate of form:

$$\mathbb{E}[\|v(\cdot,t)-u(\cdot,t)\|] \le C(\mathbf{A},\mathbf{B},\mathbf{F},\mathbf{G},\sigma,\tau,u_0,v_0,t)M(\mathbf{B}-\mathbf{A},\mathbf{G}-\mathbf{F},\tau-\sigma,v_0-u_0,t),$$

where M tends to zero as the arguments $(\mathbf{B}-\mathbf{A},\mathbf{G}-\mathbf{F},\tau-\sigma,v_0-u_0)$ tend to zero, and $\|\cdot\|$ is a norm or semi-norm.

To prove the full continuous dependence estimates, we use our (fractional) BV estimates to refine both the mollification estimates (3.46) and the estimates in Proposition 3.2.

Theorem 6.1. *Let u be a stochastic kinetic solution of* (3.1) *on $\mathbb{T}^d$ with initial data $u_0\in N^{\kappa,1}\cap L^p$ for $\kappa\in[\kappa_{F2},1]$. Let v be a stochastic kinetic solution of* (3.2) *on $\mathbb{T}^d$ with initial data $v_0\in L^p$. Assume that $\mathbf{F}$ and $\mathbf{G}$ satisfy* (1.6)–(1.8) *and* (3.3)–(3.5)*, respectively. Let σ and τ satisfy* (1.9) *and* (3.6) *with $\lambda_\sigma,\lambda_\tau>\frac{1}{2}$, and let $\mathbf{A}$ and $\mathbf{B}$ satisfy* (1.10) *and* (3.7) *with $\gamma_{\boldsymbol{\alpha}},\gamma_{\boldsymbol{\beta}}>\frac{1}{2}$, respectively. For any real constants $\rho,\theta>0$, the following continuous dependence estimate holds*:

$$\begin{aligned}
&\mathbb{E}\Big[\int(v(\mathbf{x},t)-u(\mathbf{x},t))_+\,\mathrm{d}\mathbf{x}\Big]\\
&\le C\rho+C\theta^{\kappa_{F2}}\exp\big\{C\|D_{\mathbf{x}}\cdot\mathbf{F}\|_{L^\infty}t\big\}\big(\hat{K}_1(\mathbf{F},t)+\mathbb{E}\big[|u_0|_{N^{\kappa,1}}\big]\big)\\
&\quad+\exp\{\mathcal{L}t\}\Big(\mathbb{E}\Big[\iint\eta_\rho(v(\mathbf{y},0)-u(\mathbf{x},0))J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\Big]\\
&\qquad\qquad+\big(\rho^{\kappa_{F1}}\theta^{-1}+\theta^{\kappa_{F2}}\big)\hat{K}(u_0,v_0,t)+Ct\rho^{-1}\big(\rho^{2\lambda_\sigma}+\|\tau-\sigma\|^2_{L^\infty}\big)\Big),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{L}=&\,C(\boldsymbol{\alpha})\big(\|\sqrt{\mathbf{B}}-\sqrt{\mathbf{A}}\|^2_{L^\infty}+\rho^{2\gamma_{\boldsymbol{\alpha}}}\big)\theta^{-2}\\
&+C\big(\|\mathbf{G}_u-\mathbf{F}_u\|_{L^\infty}\theta^{-1}+\|D_{\mathbf{x}}\cdot(\mathbf{G}-\mathbf{F})\|_{L^\infty}\rho^{-1}\big)+C\|D_{\mathbf{x}}\cdot\mathbf{F}_u\|_{L^\infty},
\end{aligned} \tag{6.1}$$

with all the constituent differences assumed to be bounded,

$$\hat{K}(u_0, v_0, t) = E\big[\int_0^t \big(\|(u,v)(\cdot,s)\|_{L^p}^p + 1\big)\,\mathrm{d}s\big]$$
$$\leq \exp\big\{C_0 T\,\mathbb{E}[\|(u_0,v_0)\|_{L^p}^p]\big\} \qquad \text{for } t \in [0,T],$$

and C_0 *is a constant depending on* $\mathbf{F}, \mathbf{G}, \mathbf{A}, \mathbf{B}, \sigma, \tau, d, T,$ *and* $|\mathbb{T}^d|$.

In particular, for $u_0 \in BV \cap L^p$ *and* $\kappa_{F2} = 1$*, we can choose* $\mu < \kappa_{F1}$ *and set*

$$\rho^\mu = \theta = t^{\frac{\mu}{2}}\big(\big\|(\mathbf{G}_u - \mathbf{F}_u, \sqrt{\mathbf{B}} - \sqrt{\mathbf{A}})\big\|_{L^\infty} + \big\|(\tau - \sigma, D_{\mathbf{x}}\cdot(\mathbf{G}-\mathbf{F}))\big\|_{L^\infty}^\mu\big) \quad (6.2)$$

to yield that there exists a constant $C > 0$*, depending on* $T > 0$*, such that*

$$\mathbb{E}\big[\int (v(\mathbf{x},t) - u(\mathbf{x},t))_+\,\mathrm{d}\mathbf{x}\big]$$
$$\leq C\mathbb{E}\big[\int (v_0(\mathbf{x}) - u_0(\mathbf{x}))_+\,\mathrm{d}\mathbf{x}\big]$$
$$+ C\Big(\big\|(\mathbf{G}_u - \mathbf{F}_u, \sqrt{\mathbf{B}} - \sqrt{\mathbf{A}})\big\|_{L^\infty} + \big\|(\tau - \sigma, D_{\mathbf{x}}\cdot(\mathbf{G}-\mathbf{F}))\big\|_{L^\infty}^\mu\Big)^r$$

for any $0 < \mu < \kappa_{F1}$*, where* $r := \min\{\frac{\kappa_{F1}}{\mu} - 1,\ \frac{2\lambda_\sigma - 1}{\mu},\ 1,\ \frac{1}{\mu}\}$.

PROOF. We divide the proof into three steps.

1. *Refinement of the mollification estimate*: With the assumption that $u_0 \in N^{\kappa,1}$, we return to the mollification estimate (3.46) and (5.3):

$$\mathbb{E}[I^\eta] \leq C\theta^{\kappa_{F2}} \exp\big\{C\|D_{\mathbf{x}}\cdot\mathbf{F}\|_{L^\infty} t\big\}\big(\hat{K}_1(\mathbf{F},t) + \mathbb{E}\big[|u_0|_{N^{\kappa_{F2},1}}\big]\big). \qquad (6.3)$$

Moreover, when $t = 0$,

$$\mathbb{E}\big[\big|\iint \eta_\rho(v_0(\mathbf{y}) - u_0(\mathbf{x}))J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} - \int (v_0(\mathbf{y}) - u_0(\mathbf{y}))_+\,\mathrm{d}\mathbf{y}\big|\big]$$
$$\leq C\rho + C\theta^{\kappa_{F2}}\mathbb{E}\big[|u_0|_{N^{\kappa_{F2},1}}\big].$$

2. *Continuous dependence estimate*: We now prove the general continuous dependence estimate for the initial data in $N^{\kappa_{F2},1}$. From Proposition 3.2, we

have the estimates:

$$
\begin{aligned}
&\mathbb{E}\big[\iint \eta_\rho(v(\mathbf{y},t)-u(\mathbf{x},t))J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\big]\\
&\le \mathbb{E}\big[\iint \eta_\rho(v_0(\mathbf{y})-u_0(\mathbf{x}))J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\big]\\
&\quad + C\|D_\mathbf{x}\cdot\mathbf{F}_u\|_{L^\infty}\mathbb{E}\big[\int_0^t\iint \eta_\rho(v-u)J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s\big]\\
&\quad + \Big(C(\boldsymbol{\alpha})\theta^{-2}\big(\|\sqrt{\mathbf{B}}-\sqrt{\mathbf{A}}\|_{L^\infty}^2+\rho^{2\gamma_{\boldsymbol{\alpha}}}\big)\\
&\qquad + C\big(\|\mathbf{G}_u-\mathbf{F}_u\|_{L^\infty}\theta^{-1}+\|D_\mathbf{x}\cdot(\mathbf{G}-\mathbf{F})\|_{L^\infty}\rho^{-1}\big)\Big)\\
&\qquad\quad \times\mathbb{E}\big[\int_0^t\iint \eta_\rho(v-u)J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s\big]\\
&\quad + Ct\big(\rho^{-1}\|\sigma-\tau\|_{L^\infty}^2+\rho^{2\lambda_\sigma-1}\big)|\mathbb{T}^d| + \big(\rho^{\kappa_{F1}}\theta^{-1}+\theta^{\kappa_{F2}}\big)\hat{K}(u_0,v_0,t).
\end{aligned}
$$

Applying the Gronwall inequality to the preceding calculation, we have

$$
\begin{aligned}
&\mathbb{E}\big[\iint \eta_\rho(v(\mathbf{y},t)-u(\mathbf{x},t))J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\big]\\
&\le e^{\mathcal{L}t}\Big(\mathbb{E}\big[\iint \eta_\rho(v(\mathbf{y},0)-u(\mathbf{x},0))J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\big]\\
&\qquad + \big(\rho^{\kappa_{F1}}\theta^{-1}+\theta^{\kappa_{F2}}\big)\hat{K}(u_0,v_0,t) + Ct\big(\rho^{2\lambda_\sigma-1}+\rho^{-1}\|\tau-\sigma\|_{L^\infty}^2\big)\Big),
\end{aligned}
$$

where $\mathcal{L}$ is defined by (6.1). Now we apply the mollification estimate (6.3) to obtain

$$
\begin{aligned}
&\mathbb{E}\big[\int (v(\mathbf{x},t)-u(\mathbf{x},t))_+\,\mathrm{d}\mathbf{x}\big]\\
&\le C\rho + C\theta^{\kappa_{F2}}\exp\big\{C\|D_\mathbf{x}\cdot\mathbf{F}\|_{L^\infty}t\big\}\big(\hat{K}_1(\mathbf{F},t)+\mathbb{E}\big[|u_0|_{N^{\kappa_{F2},1}}\big]\big)\\
&\quad + \exp\{\mathcal{L}t\}\Big(\mathbb{E}\big[\iint \eta_\rho(v(\mathbf{y},0)-u(\mathbf{x},0))J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\big]\\
&\qquad + C\big(\rho^{\kappa_{F1}}\theta^{-1}+\theta^{\kappa_{F2}}\big)\hat{K}(u_0,v_0,t) + Ct\rho^{-1}\big(\rho^{2\lambda_\sigma}+\|\tau-\sigma\|_{L^\infty}^2\big)\Big).
\end{aligned}
$$

3. *Refinement of the continuous dependence estimate*: Next, we consider the BV case. Assuming that $\kappa_{F2}=1$, we can further refine the estimate.

Since $\mathbb{E}\big[|u(\cdot,t)|_{BV}\big]$ is now bounded, we can refine the estimates in Proposition 3.2. Let $P\in L^\infty$ be some generic placeholder. Then integrating by parts yields

$$\begin{aligned}
&\left|\int H(\xi-u)\bar{H}(\zeta-v)P(\xi,\zeta)\cdot\nabla_{\mathbf{x}}\varphi(\xi,\zeta,\mathbf{x},\mathbf{y})\,\mathrm{d}E\right| \\
&=\left|\int H(\xi-u)\bar{H}(\zeta-v)P(\xi,\zeta)\cdot\nabla_{\mathbf{x}}J_\theta(\mathbf{y}-\mathbf{x})\eta_\rho''(\zeta-\xi)\,\mathrm{d}E\right| \\
&=\left|\iint\int \bar{H}(\zeta-v)\nabla_{\mathbf{x}}u\cdot P(u,\zeta)J_\theta(\mathbf{y}-u)\eta_\rho''(\zeta-\xi)\,\mathrm{d}\zeta\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\right| \\
&\le \|P\|_{L^\infty}\iint \eta_\rho'(v-u)J_\theta(\mathbf{y}-\mathbf{x})|\nabla_{\mathbf{x}}u|\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&\le |\mathbb{T}^d|\|P\|_{L^\infty}|u(t)|_{BV},
\end{aligned}$$

where we have used $\eta_\rho''\ge 0$, $J_\theta\ge 0$, and the boundedness of η_ρ'. This means that

$$\theta^{-1}\iint \eta_\rho(v-u)J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}$$

in (3.51)–(3.53) can be replaced by $|\mathbb{T}^d||u(t)|_{BV}$ to avoid an application of the Gronwall inequality (which puts θ^{-1} in an exponent) and an exponential penalization in time here (which comes from estimate (5.3) on $|u(t)|_{N^{\kappa_{F2},1}}$ instead).

In particular, we have

$$\begin{aligned}
&\left|\int_0^t\int \bar{H}(\zeta-v)H(\xi-u)\left(\boldsymbol{\beta}(\zeta)-\boldsymbol{\alpha}(\zeta)\right)\left(\boldsymbol{\beta}(\zeta)-\boldsymbol{\alpha}(\zeta)\right):\nabla_{\mathbf{x}}^2\varphi\,\mathrm{d}E\mathrm{d}s\right| \\
&\qquad\le d\left\|\sqrt{\mathbf{B}}-\sqrt{\mathbf{A}}\right\|_{L^\infty}^2\theta^{-1}\int_0^t|u(\cdot,s)|_{BV}\,\mathrm{d}s, \qquad (6.4)
\end{aligned}$$

$$\begin{aligned}
&\int_0^t\int \bar{H}(\zeta-v)H(\xi-u)\left(\boldsymbol{\alpha}(\zeta)-\boldsymbol{\alpha}(\xi)\right)\left(\boldsymbol{\beta}(\zeta)-\boldsymbol{\alpha}(\zeta)\right):\nabla_{\mathbf{x}}^2\varphi\,\mathrm{d}E\mathrm{d}s \\
&\qquad\le C(\boldsymbol{\alpha},d)\left\|\sqrt{\mathbf{B}}-\sqrt{\mathbf{A}}\right\|_{L^\infty}\rho^{\gamma_{\boldsymbol{\alpha}}}\theta^{-1}\int_0^t|u(\cdot,s)|_{BV}\,\mathrm{d}s, \qquad (6.5)
\end{aligned}$$

$$\begin{aligned}
&\int_0^t\int \bar{H}(\zeta-v)H(\xi-u)\left(\boldsymbol{\alpha}(\xi)-\boldsymbol{\alpha}(\zeta)\right)\left(\boldsymbol{\alpha}(\xi)-\boldsymbol{\alpha}(\zeta)\right):\nabla_{\mathbf{x}}^2\varphi\,\mathrm{d}E\mathrm{d}s \\
&\qquad\le C(\boldsymbol{\alpha},d)\rho^{2\gamma_{\boldsymbol{\alpha}}}\theta^{-1}\int_0^t|u(\cdot,s)|_{BV}\,\mathrm{d}s, \qquad (6.6)
\end{aligned}$$

in place of (3.51)–(3.53).

Similarly, we have

$$\begin{aligned}
&\left|\int_0^t\int \bar{H}(\zeta-v)H(\xi-u)\big(\mathbf{G}_u(\xi,\mathbf{x})-\mathbf{F}_u(\xi,\mathbf{x})\big)\cdot\nabla_{\mathbf{x}}\varphi\,\mathrm{d}E\mathrm{d}s\right| \\
&\le C\|\mathbf{G}_u-\mathbf{F}_u\|_{L^\infty}\int_0^t|u(\cdot,s)|_{BV}\,\mathrm{d}s, \qquad (6.7)
\end{aligned}$$

in place of (3.56).

As in Step 2 above, using Proposition 3.2, we arrive at the bound:

$$
\begin{aligned}
&\mathbb{E}\big[\iint \eta_\rho(v(\mathbf{y},t)-u(\mathbf{x},t))J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\big]\\
&\le \mathbb{E}\big[\iint \eta_\rho(v_0(\mathbf{y})-u_0(\mathbf{x}))J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\big]\\
&\quad + C\|D_{\mathbf{x}}\cdot\mathbf{F}_u\|_{L^\infty}\mathbb{E}\big[\int_0^t\iint \eta_\rho(v-u)J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s\big]\\
&\quad + C(\boldsymbol{\alpha})\big(\|\sqrt{\mathbf{B}}-\sqrt{\mathbf{A}}\|_{L^\infty}^2\theta^{-1}+\rho^{2\gamma_{\boldsymbol{\alpha}}}\theta^{-1}+\|\mathbf{G}_u-\mathbf{F}_u\|_{L^\infty}\big)\mathbb{E}\big[\int_0^t |u(\cdot,s)|_{BV}\,\mathrm{d}s\big]\\
&\quad + C\|D_{\mathbf{x}}\cdot(\mathbf{G}-\mathbf{F})\|_{L^\infty}\rho^{-1}\mathbb{E}\big[\int_0^t\iint \eta_\rho(v-u)J_\theta(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}s\big]\\
&\quad + Ct\big(\rho^{-1}\|\tau-\sigma\|_{L^\infty}^2+\rho^{2\lambda_\sigma-1}\big)|\mathbb{T}^d| + C\big(\rho^{\kappa_{F1}}\theta^{-1}+\theta^{\kappa_{F2}}\big)\hat{K}(u_0,v_0,t).
\end{aligned}
$$

Estimating the mollification and BV terms, we derive a continuous dependence estimate for

$$\mathbb{E}\big[\iint \eta_\rho(v(\mathbf{x},t)-u(\mathbf{x},t))\,\mathrm{d}\mathbf{x}\big]$$

as before. Since $2\gamma_{\boldsymbol{\alpha}}>1$, we can choose $\mu<\kappa_{F1}$ and set ρ and θ as in (6.2) to complete the proof.

Remark 6.1. Estimate (6.7) can also be applied to (3.59) if $\mathbf{F}_u(\xi,\mathbf{x})-\mathbf{F}_u(\zeta,\mathbf{x})$ is assumed to be uniformly bounded, replacing θ^{-1} by $\int_0^t|u(\cdot,s)|_{BV}\,\mathrm{d}s$ in (3.59).

Remark 6.2. The case that $\mathbf{A}$ depends on $\mathbf{x}$, *i.e.*, $\mathbf{A}=\mathbf{A}(u,\mathbf{x})$, behaves differently, and additional difficulties present themselves. In particular, for the BV–estimate, in order to make a sense of the calculations, one might take the ith derivative of the entire equation (at the bulk, non-kinetic level) and test it against $\eta'_\rho(\partial_i u)$. One cannot easily propose an assumption on $\mathbf{A}(u,\mathbf{x})_{x_i}$ by which to bound the terms:

$$\int \eta''_\rho(u_{x_i})\mathbf{A}(u,\mathbf{x})_{x_i}:(\nabla u\otimes\nabla u_{x_i})\,\mathrm{d}\mathbf{x},$$

since the second derivatives inevitably appear in the estimates.

7. Existence of Stochastic Kinetic Solutions

In this section, we employ the continuous dependence estimate to establish the existence of stochastic kinetic solutions. In order to achieve proper energy estimates in $L^p(\mathbb{T}^d)$, we require the assumption that

$$|D_{\mathbf{x}}\cdot\mathbf{F}(u,\mathbf{x})|\le C\big(|u|+1\big).$$

Remark 7.1. With reference to Remark 3.1, it is possible to extend this result to $L^p(\mathbb{R}^d)$, since only the L^1–stability is actually used.

7.1. Convergence in ε

Let u_0^ε be a collection of the initial data functions that tend to u_0 in $L^1_\omega L^1_{\mathbf{x}}$.

We show here that there is a subsequence of the corresponding viscosity kinetic solutions u^ε (see Appendix A for the well-posedness of solutions with almost surely continuous paths in $L^p(\mathbb{T}^d)$), which converges to a unique stochastic kinetic solution. From the continuous dependence estimates, we conclude that the kinetic solutions u^ε and $u^{\varepsilon'}$ of

$$\partial_t u^\varepsilon = -\nabla\cdot\mathbf{F}(u^\varepsilon,\mathbf{x}) + \nabla\cdot\big((\mathbf{A}(u^\varepsilon)+\varepsilon\mathbf{I})\nabla u^\varepsilon\big) + \sigma(u^\varepsilon)\dot{W},$$

and

$$\partial_t u^{\varepsilon'} = -\nabla\cdot\mathbf{F}(u^{\varepsilon'},\mathbf{x}) + \nabla\cdot\big((\mathbf{A}(u^{\varepsilon'})+\varepsilon'\mathbf{I})\nabla u^{\varepsilon'}\big) + \sigma(u^{\varepsilon'})\dot{W},$$

satisfy

$$\begin{aligned}
&\mathbb{E}\Big[\int |u^{\varepsilon'}(\mathbf{x},t)-u^\varepsilon(\mathbf{x},t)|\,\mathrm{d}\mathbf{x}\Big]\\
&\le C\Big(\mathbb{E}\Big[\int |u_0^{\varepsilon'}(\mathbf{x})-u_0^\varepsilon(\mathbf{x})|\,\mathrm{d}\mathbf{x}\Big] + |\sqrt{\varepsilon}-\sqrt{\varepsilon'}|^{\min\{\frac{\kappa_{F1}}{\mu}-1,\,\frac{2\lambda_\sigma-1}{\mu},\,\kappa_{F2},\,\frac{1}{\mu}\}}\Big)
\end{aligned}$$

for $0<\mu<\kappa_{F1}$. Then we conclude that

$$\mathbb{E}\big[\|u^{\varepsilon'}(\mathbf{x},t)-u^\varepsilon(\mathbf{x},t)\|_{L^1(\mathbb{T}^d\times[0,T])}\big]\to 0 \qquad \text{as } \varepsilon,\varepsilon'\to 0. \tag{7.1}$$

Moreover, including the martingale part in order to estimate the difference in the uniform norm in time, we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}\int |u^{\varepsilon'}(\mathbf{x},t)-u^\varepsilon(\mathbf{x},t)|\,\mathrm{d}\mathbf{x}\Big]\\
&\le C\Big(\mathbb{E}\Big[\int |u_0^{\varepsilon'}(\mathbf{x})-u_0^\varepsilon(\mathbf{x})|\,\mathrm{d}\mathbf{x}\Big] + |\sqrt{\varepsilon}-\sqrt{\varepsilon'}|^{\min\{\frac{\kappa_{F1}}{\mu}-1,\,\frac{2\lambda_\sigma-1}{\mu},\,\kappa_{F2},\,\frac{1}{\mu}\}}\Big)\\
&\quad + \mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t\int \big(\sigma(u^\varepsilon)-\sigma(u^{\varepsilon'})\big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}W\Big|\Big].
\end{aligned} \tag{7.2}$$

By the Burkholder–Davis–Gundy inequality and Young's inequality,

$$
\begin{aligned}
&\mathbb{E}\big[\sup_{t\in[0,T]}\Big|\int_0^t\int\big(\sigma(u^\varepsilon)-\sigma(u^{\varepsilon'})\big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}W\Big|\big]\\
&C\le\mathbb{E}\Big[\Big|\int_0^T\int\frac{\big|\sigma(u^\varepsilon)-\sigma(u^{\varepsilon'})\big|^2}{|u^\varepsilon-u^{\varepsilon'}|^{2\lambda_\sigma}}|u^\varepsilon-u^{\varepsilon'}|^{2\lambda_\sigma}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\Big|^{1/2}\Big]\\
&\le C_\sigma\mathbb{E}\Big[\Big|\int_0^T\int|u^\varepsilon-u^{\varepsilon'}|^{2\lambda_\sigma}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\Big|^{1/2}\Big]\\
&\le C_\sigma\mathbb{E}\big[\int_0^T\int|u^\varepsilon-u^{\varepsilon'}|^{2\lambda_\sigma-1}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\big]+\frac{1}{2}\mathbb{E}\big[\sup_{t\in[0,T]}\int|u^\varepsilon-u^{\varepsilon'}|\,\mathrm{d}\mathbf{x}\big]\\
&\le C_{\sigma,T}\Big(\mathbb{E}\big[\int_0^T\int|u^\varepsilon-u^{\varepsilon'}|\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\big]\Big)^{2\lambda_\sigma-1}+\frac{1}{2}\mathbb{E}\big[\sup_{t\in[0,T]}\int|u^\varepsilon-u^{\varepsilon'}|\,\mathrm{d}\mathbf{x}\big].
\end{aligned}
$$

The final inequality is the result of Jensen's inequality, as $2\lambda_\sigma-1\in(0,1)$.

Since $2\lambda_\sigma-1>0$, from (7.2), we have the following bound:

$$
\begin{aligned}
&\frac{1}{2}\mathbb{E}\big[\sup_{t\in[0,T]}\int|u^{\varepsilon'}(\mathbf{x},t)-u^\varepsilon(\mathbf{x},t)|\,\mathrm{d}\mathbf{x}\big]\\
&\le C\Big(\mathbb{E}\big[\int|u_0^{\varepsilon'}(\mathbf{x})-u_0^\varepsilon(\mathbf{x})|\,\mathrm{d}\mathbf{x}\big]+|\sqrt{\varepsilon}-\sqrt{\varepsilon'}|^{\min\{\frac{\kappa_{F1}}{\mu}-1,\,\frac{2\lambda_\sigma-1}{\mu},\,\kappa_{F2},\,\frac{1}{\mu}\}}\Big)\\
&\quad+f(\varepsilon,\varepsilon'),
\end{aligned}
$$

where

$$
f(\varepsilon,\varepsilon'):=C_{\sigma,T}\Big(\mathbb{E}\big[\int_0^T\int|u^\varepsilon-u^{\varepsilon'}|\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\big]\Big)^{2\lambda_\sigma-1}\to0
$$

as $\varepsilon,\varepsilon'\to0$, by (7.1).

Finally, we conclude that

$$
\mathbb{E}\big[\sup_{t\in[0,T]}\|u^{\varepsilon'}(\mathbf{x},t)-u^\varepsilon(\mathbf{x},t)\|_{L^1(\mathbb{T}^d)}\big]\to0\qquad\text{as }\varepsilon,\varepsilon'\to0.
$$

That is, the approximate solution sequence $\{u^\varepsilon\}$ is a Cauchy sequence in $L^1_\omega C_t L^1_{\mathbf{x}}$ so that there is a subsequence (still denoted) $\{u^\varepsilon\}$ converging to a process u with almost surely continuous paths in $L^1(\mathbb{T}^d)$.

7.2. Existence Theorem

With the convergence of $\{u^\varepsilon\}$ obtained in §7.1, we can follow [5] to conclude the following existence theorem:

Theorem 7.1 (Existence of stochastic kinetic solutions)**.** *Let assumptions* (1.6)–(1.10) *hold. Then there exists a unique stochastic kinetic solution of equation* (1.1) *with initial data $u_0\in L^1$. In particular, if initial data $u_0\in L^p\cap N^{\kappa,1}$, then the stochastic kinetic solution $u(\cdot,t)\in L^p\cap N^{\kappa,1}$ for each $t>0$.*

Proof. For any fixed ε, we can mollify u_0 into $u_0^\varepsilon \in C^\infty$ so that $\mathbb{E}[\|u_0^\varepsilon\|_{H^s}^2]$ is bounded for any s and

$$\mathbb{E}\big[\|u_0^\varepsilon\|_{L^p}^p\big] \leq C\,\mathbb{E}\big[\|u_0\|_{L^p}^p\big] < \infty,$$

where $C > 0$ is a constant independent of $\varepsilon > 0$. Then, as in [5], using the arguments of §4 of Feng–Nualart [17], together with the convergence results in §7.1, we can conclude that there is a convergent subsequence $u^\varepsilon(\mathbf{x}, t)$ that converges *a.e.* almost surely to $u(\mathbf{x}, t)$ that is a stochastic kinetic solution. The L^1–stability of stochastic kinetic solutions implies the uniqueness of the solution.

In particular, if $\mathbb{E}\big[\|u_0\|_{L^p}^p\big] + \mathbb{E}\big[|u_0|_{N^{\kappa,1}}\big] < \infty$, by the continuous dependence estimates, we conclude that

$$\sup_{t>0}\big(\mathbb{E}\big[\|u(\cdot,t)\|_{L^p}^p\big] + \mathbb{E}\big[|u(\cdot,t)|_{N^{\kappa,1}}\big]\big) < \infty.$$

8. Temporal Fractional BV Regularity of Stochastic Kinetic Solutions

In this section, we prove that the stochastic kinetic solution is of fractional BV regularity in time.

Theorem 8.1. *Let $u(\mathbf{x}, t)$ be a stochastic kinetic solution of* (1.1)–(1.2) *with initial data $u_0(\cdot) \in L^p \cap N^{\kappa_1,1}$ for some $\kappa_1 \in (0, 1]$. Let $D_\mathbf{x} \cdot \mathbf{F}$ have linear growth in u and κ_2–Hölder in $\mathbf{x}$ for some $\kappa_2 > 0$. Let σ have linear growth, and let the entries of $\mathbf{A}$ have polynomial growth in u. Then there exists $\beta > 0$ depending on κ_1 and κ_2 such that, for any $T > 0$, there is $C_T > 0$ so that*

$$\mathbb{E}\Big[\int_0^{T-\Delta t}\int (u(\mathbf{x}, t+\Delta t) - u(\mathbf{x}, t))_+ \,\mathrm{d}\mathbf{x}\mathrm{d}t\Big] \leq C_T(\Delta t)^\beta \qquad \textit{for any } \Delta t \in (0, 1).$$

Proof. Define the temporal difference:

$$w(\cdot, t) := u(\cdot, t+\Delta t) - u(\cdot, t).$$

From the definition of stochastic kinetic solutions, for a test function $\varphi(\xi, \mathbf{x}, t)$, we have

$$
\begin{aligned}
&\langle \bar{H}(\xi - u(\cdot, t+\Delta t)), \varphi\rangle - \langle \bar{H}(\xi - u(\cdot, t)), \varphi\rangle \\
&= \int_t^{t+\Delta t} \langle \bar{H}(\cdot - u(\cdot, s))\, \mathbf{F}_u, \nabla\varphi\rangle \,\mathrm{d}s - \int_t^{t+\Delta t} \langle\, \bar{H}(\cdot - u(\cdot, s))\, D_{\mathbf{x}} \cdot \mathbf{F}, \varphi_\xi\rangle \,\mathrm{d}s \\
&\quad + \int_t^{t+\Delta t} \langle \bar{H}(\cdot - u(\cdot, s)), \mathbf{A} : \nabla^2\varphi\rangle \,\mathrm{d}s - \int_t^{t+\Delta t} \iint \varphi_\xi \,\mathrm{d}(m^u + n^u)(\xi, \mathbf{x}, s) \\
&\quad + \frac{1}{2}\int_t^{t+\Delta t} \int \sigma^2(u(\mathbf{x}, s))\varphi_\xi(u(\mathbf{x}, s), \mathbf{x}, t) \,\mathrm{d}\mathbf{x}\mathrm{d}s \\
&\quad + \int_t^{t+\Delta t} \int \sigma(u(\mathbf{x}, s))\varphi(u(\mathbf{x}, s), \mathbf{x}) \,\mathrm{d}\mathbf{x}\mathrm{d}W(s),
\end{aligned}
$$

where, as in (3.16), the angle brackets represent the integrals in $(\mathbf{x}, \xi)$. As before, $\bar{H} := 1 - H$ with H as the Heaviside function.

We now choose a test function that is monotonically increasing in the kinetic variable ξ, so that we can avail ourselves of the sign of the defect measures in the effort to estimate the left-hand side. We retain the positive part function in favor of the sign function.

Nevertheless, inspired by [5], we use the test function:

$$
\varphi(\xi, \mathbf{x}, t) = (J_\theta * (\mathrm{sgn}(w(\cdot, t)))_+)(\mathbf{x})\, \eta'_\rho(\xi - u(\mathbf{x}, t)) \geq 0,
$$

where J_θ is again an approximation to $\delta_0(\mathbf{x})$ that is a smooth non-negative function with support on $B_\theta(\mathbf{0})$ and unit mass. Let $\eta_\rho : \mathbb{R} \to \mathbb{R}$ continue to be as in the construction given in (3.9).

Integrating above from 0 to $T-\Delta t$ in t, we have the expression:

$$
\begin{aligned}
&\int_0^{T-\Delta t} \langle \bar{H}(\xi - u(\cdot, t+\Delta t)) - \bar{H}(\xi - u(\cdot, t)),\, \varphi\rangle\, \mathrm{d}t \\
&= \int_0^{T-\Delta t}\int_t^{t+\Delta t} \langle \bar{H}(\cdot - u(\cdot, s))\, \mathbf{F}_u, \nabla\varphi\rangle\, \mathrm{d}s\mathrm{d}t \\
&\quad - \int_0^{T-\Delta t}\int_t^{t+\Delta t} \langle \bar{H}(\cdot - u(\cdot, s))\, D_{\mathbf{x}}\cdot \mathbf{F},\, \varphi_\xi\rangle\, \mathrm{d}s\mathrm{d}t \\
&\quad + \int_0^{T-\Delta t}\int_t^{t+\Delta t} \langle \bar{H}(\cdot - u(\cdot, s)),\, \mathbf{A} : \nabla^2\varphi\rangle\, \mathrm{d}s\mathrm{d}t \\
&\quad - \int_0^{T-\Delta t}\int_t^{t+\Delta t} \iint \varphi_\xi\, \mathrm{d}(m^u + n^u)(\xi, \mathbf{x}, s)\; \mathrm{d}t \\
&\quad + \frac{1}{2}\int_0^{T-\Delta t}\int_t^{t+\Delta t} \int \sigma^2(u(\mathbf{x}, s))\varphi_\xi(u(\mathbf{x}, s), \mathbf{x}, t)\; \mathrm{d}\mathbf{x}\mathrm{d}s\mathrm{d}t \\
&\quad + \int_0^{T-\Delta t}\int_t^{t+\Delta t} \int \sigma(u(\mathbf{x}, s))\varphi(u(\mathbf{x}, s), \mathbf{x}, t)\; \mathrm{d}\mathbf{x}\mathrm{d}W(s)\mathrm{d}t \\
&\quad + \int_0^{T-\Delta t} \langle \bar{H}(\xi - u(\cdot, t+\Delta t)) - |w(\cdot, t)|,\, \varphi\rangle\, \mathrm{d}t.
\end{aligned}
\tag{8.1}
$$

Notice that, though the test function φ depends on $u(\cdot, t+\Delta t)$, one can integrate first in s in the stochastic integral above so that all the integrals are adapted and well-defined either in the Lebesgue–Stieljes sense or, more generally, in the Itô sense.

On the left-hand side of (8.1), from the presence of $\eta'_\rho(\xi - u(\cdot, t))$ in the definition of φ, we expect that $\langle H(\xi - u(\cdot, t)), \varphi\rangle \to 0$ as $\rho \to 0$. We have the following estimate:

$$
\left| \int_0^{T-\Delta t} \langle \bar{H}(\xi - u(\cdot, t)), \varphi\rangle\, \mathrm{d}t \right| \le C_T \rho.
$$

For the right-hand side of (8.1), we first note that, as remarked previously,

$$
\int_t^{t+\Delta t} \iint \varphi_\xi\, \mathrm{d}(m^u + n^u)(\xi, \mathbf{x}, s) \ge 0.
$$

We proceed to analyze the remaining parts of the right-hand side of (8.1).

Flux terms: Since $D_{\mathbf{x}}\cdot \mathbf{F}$ has linear growth in u, then the L^p estimate of u

implies that

$$
\begin{aligned}
&\Big|\mathbb{E}\big[\int_0^{T-\Delta t}\int_t^{t+\Delta t}\langle \bar{H}(\cdot-u(\cdot,s))\,\mathbf{F}_u,\nabla\varphi\rangle\,\mathrm{d}s\mathrm{d}t\big]\\
&\;-\mathbb{E}\big[\int_0^{T-\Delta t}\int_t^{t+\Delta t}\langle \bar{H}(\cdot-u(\cdot,s))\,D_{\mathbf{x}}\cdot\mathbf{F},\,\varphi_\xi\rangle\,\mathrm{d}s\mathrm{d}t\big]\Big|\\
&\le C_T\big(\theta^{-1}\Delta t+\rho^{-1}\Delta t\big).
\end{aligned}
$$

Parabolic term: Using the polynomial growth of the entries of $\mathbf{A}$, we have

$$
\Big|\mathbb{E}\big[\int_0^{T-\Delta t}\int_t^{t+\Delta t}\langle \bar{H}(\cdot-u(\cdot,s)),\,\mathbf{A}:\nabla^2\varphi\rangle\,\mathrm{d}s\mathrm{d}t\big]\Big|\le C_T\theta^{-2}\Delta t.
$$

Itô Correction term: Using the linear growth of σ, the L^p estimate of u implies that

$$
\frac{1}{2}\Big|\mathbb{E}\big[\int_0^{T-\Delta t}\int_t^{t+\Delta t}\int\sigma^2(u(\mathbf{x},s))\,\varphi_\xi(u(\mathbf{x},s),\mathbf{x};t)\,\mathrm{d}\mathbf{x}\mathrm{d}s\mathrm{d}t\big]\Big|\le C_T\rho^{-1}\Delta t.
$$

Noise term: Using the Burkholder–Davis–Gundy inequality and the L^p estimate of u yield

$$
\Big|\mathbb{E}\big[\int_0^{T-\Delta t}\int_t^{t+\Delta t}\int\sigma(u(\mathbf{x},s))\varphi(u(\mathbf{x},s),\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}W(s)\mathrm{d}t\big]\Big|\le C_T\sqrt{\Delta t}.
$$

Mollification term: Since $D_{\mathbf{x}}\cdot\mathbf{F}$ is κ_2-Hölder in $\mathbf{x}$, we use the fractional BV estimate in $\mathbf{x}$ in §5 to obtain as in Chen–Ding–Karlsen [5]:

$$
\begin{aligned}
&\Big|\mathbb{E}\big[\int_0^{T-\Delta t}\big(\langle \bar{H}(\xi-u(\cdot,t+\Delta t)),\varphi\rangle-w(\cdot,t)_+\big)\,\mathrm{d}t\big]\Big|\\
&\le\mathbb{E}\big[\int_0^{T-\Delta t}\iint J_\theta(\mathbf{x}-\mathbf{y})|w(\mathbf{x},t)-w(\mathbf{y},t)|\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}t\big]\\
&\le\mathbb{E}\big[\int_0^{T}\int J(\mathbf{z})\int|u(\mathbf{x},t)-u(\mathbf{x}-\theta\mathbf{z},t)|\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{z}\mathrm{d}t\big]\\
&\le C_T\theta^{\min\{\kappa_1,\kappa_2\}}.
\end{aligned}
$$

Conclusion: Taking $\rho=\theta^2$ and $\theta=(\Delta t)^\alpha$, we have

$$
\begin{aligned}
&\mathbb{E}\big[\int_0^{T-\Delta t}\int|w(\mathbf{x},t)|\,\mathrm{d}\mathbf{x}\mathrm{d}t\big]\\
&\le C\big((\Delta t)^{2\alpha}+(\Delta t)^{1-\alpha}+(\Delta t)^{1-2\alpha}+(\Delta t)^{\frac{1}{2}}+(\Delta t)^{\alpha\min\{\kappa_1,\kappa_2\}}\big).
\end{aligned}
$$

This allows us to optimize α to conclude that there exists β depending on κ_1

and κ_2 such that

$$\mathbb{E}\Big[\int_0^{T-\Delta t}\int |w(\mathbf{x},t)|\,\mathrm{d}\mathbf{x}\mathrm{d}t\Big] \le C|\Delta t|^{\beta}. \tag{8.2}$$

This completes the proof.

Remark 8.1. In [5], it is conjectured that the optimal bound for the first-order conservation law is $(\Delta t)^{\frac{1}{2}}$. If the BV bound of the solution is in place of a fractional BV bound, the conjecture holds true on the torus for that case. However, in the second-order case, the presence of the second derivative provides another power of θ^{-1} in the presence of a spatial BV bound, which leads to a bound $C(\Delta t)^{\beta}$ under the optimization.

Appendix A. Existence of Solutions to the Uniformly Parabolic Equations

In this appendix, we sketch out the proof for the well-posedness of strong solutions to the stochastic parabolic equations of form:

$$\begin{cases}\partial_t u - \nabla\cdot((\varepsilon\mathbf{I}+\mathbf{A}(u))\,\nabla u) + \nabla\cdot\mathbf{F}(u,\mathbf{x}) = \sigma(u)\dot{W} \qquad \text{on } \mathbb{T}^d\times[0,T],\\ u|_{t=0}=u^0\end{cases} \tag{A.1}$$

whose coefficients satisfy the assumptions laid out in (1.6)–(1.10). This is a small extension of [21] or [17, §4] (see also [11, Ch. 3]).

From [21], we know that there is a unique strong solution to

$$\begin{cases}\partial_t u = \nabla\cdot((\varepsilon\mathbf{I}+\mathbf{A}(v))\,\nabla u) - \nabla\cdot\mathbf{F}(v,\mathbf{x}) - \sigma(v)\dot{W},\\ u|_{t=0}=u^0\end{cases} \tag{A.2}$$

for a fixed v, an adapted process in $L^p(\Omega; C([0,T]; L^p(\mathbb{T}^d)))$.

We consider the linearization:

$$\partial_t u^n = \varepsilon\Delta u^n + \nabla\cdot\big(\mathbf{A}(u^{n-1})\nabla u^n\big) - \nabla\cdot\mathbf{F}(u^{n-1},x) - \sigma(u^{n-1})\dot{W}.$$

By the Duhamel formula, we use the following iteration scheme:

$$\begin{aligned}u^n = G_{n-1}(t)*u^0 - \int_0^t G_{n-1}(t-s)*\nabla\cdot\mathbf{F}(u^{n-1},\cdot)\,\mathrm{d}s\\ - \int_0^t G_{n-1}(t-s)*\sigma(u^{n-1}(\cdot,s))\,\mathrm{d}W(s),\end{aligned}$$

where the convolution is in $\mathbf{x}$ only, and G_{n-1} is the parametrix of the parabolic equation:

$$\partial_t u - \nabla\cdot\big((\varepsilon I + \mathbf{A}(u^{n-1}))\nabla u\big) = 0.$$

Then the existence and uniqueness follow by a fixed-point argument, as in [21, §4], by using the additional direct estimates on the parametrix:

$$\|G_{n-1}(t)\|_{L^1} \leq \|G(t)\|_{L^1}, \quad \|\nabla G_{n-1}(t)\|_{L^1} \leq \|\nabla G(t)\|_{L^1} \qquad \text{uniformly in } n, \tag{A.3}$$

where G is the standard heat kernel (with $\mathbf{A} \equiv 0$). These bounds hold because, by scaling t, we see that the eigenvalues of the operator (on the compact domain $\mathbb{T}^d$) when $\mathbf{A} \equiv 0$ must be larger than those when $\mathbf{A} \geq 0$.

Consider $\mathbf{F}$ with linear growth in u. Then we can estimate $\|u\|_{L^p}^p$ by using Young's convolution inequality and Minkowski's inequality as follows:

$$\begin{aligned}\|u^n\|_{L^p} \leq \|G_{n-1}(t)\|_{L^1} \left\|u^0\right\|_{L^p} + \int_0^t \|\nabla G_{n-1}(t-s)\|_{L^1} \left\|\mathbf{F}(u^{n-1}, \cdot)\right\|_{L^p} \,\mathrm{d}s \\ + \left\| \int_0^t G_{n-1}(t-s) * \sigma(u^{n-1}(\cdot, s)) \,\mathrm{d}W \right\|_{L^p}.\end{aligned}$$

By the Burkholder–Davis–Gundy inequality, Minkowski's inequality, and Jensen's inequality, we see that

$$\begin{aligned}&\mathbb{E}[\sup_{r\in[0,t]} \left\| \int_0^r G_{n-1}(r-s) * \sigma(u^{n-1}(\cdot, s)) \,\mathrm{d}W \right\|_{L^p}^p] \\ &\leq \mathbb{E}[\left\| \int_0^t \left|G_{n-1}(t-s) * \sigma(u^{n-1}(\cdot, s))\right|^2 \,\mathrm{d}s \right\|_{L^{\frac{p}{2}}}^{\frac{p}{2}}] \\ &\leq \mathbb{E}[\left| \int_0^t \left\|G_{n-1}(t-s) * \sigma(u^{n-1}(\cdot, s))\right\|_{L^p}^2 \,\mathrm{d}s \right|^{\frac{p}{2}}] \\ &\leq t^{\frac{p}{2}-1} \mathbb{E}[\int_0^t \|G_{n-1}\|_{L^1}^p \left\|\sigma(u^{n-1})\right\|_{L^p}^p \,\mathrm{d}s].\end{aligned}$$

Next, using the fact that $\nabla_{\mathbf{x}} \cdot \mathbf{F}$ (and hence $\mathbf{F}$) has at most linear growth in u, the assumption that $\sigma(u)$ has at most linear growth in u, and (A.3), we see that, for $p \geq 2$ and a sufficiently small time t, the map: $u^{n-1} \mapsto u^n$ is a contraction on $L^p(\Omega; C([0,t]; L^p(\mathbb{T}^d)))$. The constant is independent of n, and the fixed-point argument can be iterated as usual to yield the existence and uniqueness on the whole time interval $[0,T]$ for any fixed $T > 0$. This shows that a unique strong solution exists for (A.1) in $L^p(\Omega; C([0,T]; L^p(\mathbb{T}^d)))$.

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